# Cycles and rationality of algebraic varieties

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# Abstract

This thesis studies algebraic cycles under degenerations to snc schemes and its application to rationality problems of algebraic varieties. Besides a chapter on some preliminaries, the thesis consists of four parts. The results of the first three parts appeared in the paper [LS25] as well as the preprints [Lan24] and [LZ25].

We prove in the first part that the Chow group of zero-cycles of a smooth projective variety $X$ over the Laurent series field $\mathbb{C}((t))$ is isomorphic to $\mathbb{Z}$, if $X$ admits geometrically a decomposition of the diagonal and has a strictly semi-stable model over $\mathbb{C}[[t]]$. This provides an unconditional proof in some special cases of a result Tian [Tia20], which uses the Tate conjecture for one-cycles on surfaces. We also obtain a similar result in positive characteristic.

The second part consists of joint work with Stefan Schreieder. We provide a cycle-theoretic method to disprove rationality by degenerating to snc schemes where the obstruction to rationality lies in some lower dimensional stratum. This builds on earlier work of Pavic–Schreieder [PS23]. We apply this method to obtain improved bounds for the irrationality of hypersurfaces. More precisely, we show that a very general hypersurface of degree $d \geq 4$ and dimension at most $(d+1)2^{d-4}$ over a field of characteristic different from 2 does not admit a decomposition of the diagonal, in particular it is not (retract) rational. This improves earlier work of Schreieder [Sch19b] and Moe [Moe23].

In the third part, we extend in joint work with Guoyun Zhang the above result for hypersurfaces to complete intersections and certain hypersurfaces in other rational varieties like products of projective spaces and Grassmannians. This extends earlier work of Chatzistamatiou–Levine [CL17] and Nicaise–Ottem [NO22]. The core of our argument is based on some elementary commutative algebra results.

The forth part is a small observation that algebraic equivalence behaves well under specialization in strictly semi-stable families. This might be a very first step to provide a counterexample to a question of Voisin [Voi19], whether the Griffith group of one-cycles is trivial for smooth projective rationally connected complex varieties.

# Contents

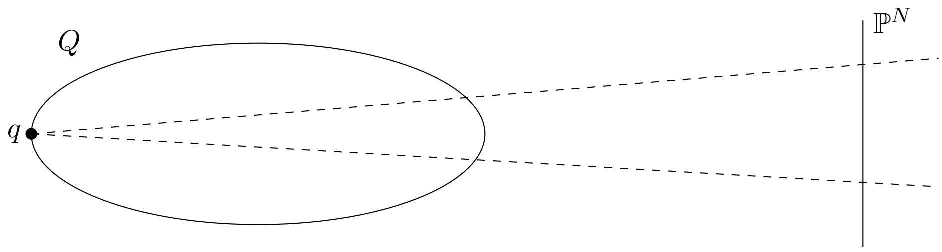
# 1. Introduction

The majority of this thesis is based on the paper [LS25] (see Chapter 4) and the preprints [Lan24] (see Chapter 3) and [LZ25] (see Chapter 5). These chapters are complemented by a brief chapter on some preliminaries (Chapter 2) as well as a small observation regarding the specialization of algebraic equivalence (Chapter 6).

Each of the main chapters contains an individual introduction; we provide here a comprehensive introduction and overview over the thesis.

## 1.1. Rationality questions

The rationality question for a given smooth projective variety $X$ over a field $k$ asks to decide whether $X$ is birational to $\mathbb{P}_k^{\dim X}$. Such a variety $X$ is called *rational* or more precisely $k$-rational. It is well known that a quadric hypersurface $Q$ in projective space $\mathbb{P}_k^{N+1}$, i.e. $Q$ is the vanishing locus of a homogeneous quadratic polynomial, is rational if and only if it contains a $k$-rational point $q \in Q(k)$. In fact, the projection from the $k$-rational point $q$ provides a birational map from the quadric hypersurface to a hyperplane $\mathbb{P}^N \subset \mathbb{P}^{N+1}$ not containing $q$:




In general the rationality question for a given smooth projective variety can be a difficult problem. For instance, no example of a smooth rational hypersurface of degree $d \geq 4$ is known, see [Kol, Question 52]. It is expected that at least the very general such hypersurface is irrational.

Birational invariants are used to distinguish irrational varieties from rational ones.

A simple birational invariant for smooth projective varieties is the $m$-th plurigenus

$$p_m(X) \coloneqq \dim_k H^0(X, \omega_X^{\otimes m})$$

for $m > 0$. The adjunction formula shows that for a smooth hypersurface $X_d$ of degree $d \geq N+2$ in $\mathbb{P}_k^{N+1}$ the $m$-th plurigenus $p_m(X_d) > 0$, in particular $X_d$ is irrational. For $d < N+2$, the hypersurface $X_d$ in $\mathbb{P}^{N+1}$ is a Fano variety (its canonical bundle is anti-ample) and all the plurigenera vanish. In characteristic 0, Fano varieties are examples of rationally connected varieties by the work of [KMM92, Cam92]: A variety over a field $k$ is called *rationally connected* if for any algebraically closed field extension $L/k$ any two very general $L$-rational points lie on an irreducible rational curve. For a smooth projective (separably)[1] rationally connected variety all plurigenera vanish, see [Kol96, Corollary IV.3.8] Thus rationally connected varieties are an interesting class of varieties for rationality questions.

Several finer notions of rationality have been introduced and studied, for instance:

- *stably rational:* $X \times \mathbb{P}_k^m$ is rational for some $m \geq 0$;
- *retract rational:* there exists a dominant rational map $\mathbb{P}_k^N \dashrightarrow X$ with a rational section;
- *unirational:* there exists a dominant rational map $\mathbb{P}_k^N \dashrightarrow X$.

The following implications are easy consequences of the above definitions:

rational $\Longrightarrow$ stably rational $\Longrightarrow$ retract rational $\Longrightarrow$ unirational $\Longrightarrow$ rationally connected.

It is known that the first and third implication are strict implications by [BCTSS85] and [AM72]. There are (geometrically rational) examples over non-closed fields which are retract rational but not stably rational, see for instance [EM75, Theorem 1.5 and Theorem 2.3]; it is an open problem to produce such examples over closed fields, see for instance [Sch25b]. As far as we know the strictness of the last implication is open over any field.

Various methods have been developed and used to study the rationality problem; we refer the reader to the survey of Schreieder [Sch25b] for an overview of recent methods. We briefly explain two methods below, which are relevant for this thesis.

[1] in positive characteristic there are rationally connected (even unirational) surfaces with non-trivial plurigenera, see [Shi74]

## 1.2. Algebraic cycles and a decomposition of the diagonal

Another (stably) birational invariant for smooth projective varieties over a field $k$ is the Chow group of zero-cycles $\mathrm{CH}_0(X)$, which is the quotient of the free abelian group generated by closed points on $X$ modulo rational equivalence. Since rationality is stable under field extension, we have for a smooth projective rational variety $X$ over a field $k$ and any field extension $L/k$

$$\mathrm{CH}_0(X_L) \simeq \mathrm{CH}_0(\mathbb{P}_L^{\dim X}) \simeq \mathbb{Z},$$

where $X_L \coloneqq X \times_k L$ is the base change. In fact, this isomorphism is the degree map for zero-cycles. A smooth projective variety $X$ over a field $k$ is called *universally* $\mathrm{CH}_0$*-trivial* if for any field extension $L/k$ the degree map

$$\deg\colon\ \mathrm{CH}_0(X_L) \longrightarrow \mathbb{Z}$$

is an isomorphism, see [ACTP17, Section 1.2]. The property of being universally $\mathrm{CH}_0$-trivial for smooth projective varieties is equivalent to having an integral decomposition of the diagonal - a notion which goes back to Bloch–Srinivas [BS83]:

Let $\Lambda$ be a commutative ring with 1. We say that a $k$-variety $X$ of dimension $n$ admits a $\Lambda$-*decomposition of the diagonal*, if the diagonal $\Delta_X \subset X \times_k X$ satisfies

$$\Delta_X = z \times X + Z \in \mathrm{CH}_n(X \times_k X, \Lambda), \tag{1.2.1}$$

where $z$ is a zero-cycle on $X$ and $Z$ is an $n$-cycle on $X \times_k X$, which does not dominate the second factor. If $\Lambda = \mathbb{Z}$, we say that $X$ admits an *integral decomposition of the diagonal.* The above discussion shows that smooth projective rational varieties admit an integral decomposition of the diagonal; in fact also retract rational varieties have an integral decomposition of the diagonal, see for instance [Sch21b, Lemma 7.5].

Voisin [Voi15] introduced a method to disprove rationality using the notion of the decomposition of the diagonal together with a degeneration argument to mildly singular varieties. Voisin showed in *loc. cit.* that a very general quartic double solid does not admit an integral decomposition of the diagonal, quartic double solids are known to be unirational. Thus the integral decomposition of the diagonal provides an obstruction to retract rationality for rationally connected varieties. The method via the decomposition of the diagonal have been improved by Colliot-Thélène–Pirutka [CTP16] and Schreieder [Sch19b]. In joint work with Schreieder

(Chapter 4), we extend this method to degenerations to snc schemes, where the obstruction to rationality lies in some lower dimensional stratum by building on earlier work of Pavic–Schreieder [PS23]. This extension is inspired by a motivic method.

## 1.3. The motivic method

A different method using the motivic volume was initiated by Nicaise–Shinder [NS19] and Kontsevich–Tschinkel [KT19], who showed that (stable) rationality specializes in smooth proper families over fields of characteristic 0, see also [NO21].

Let $F$ be a field and consider the free abelian group $\mathbb{Z}[\mathrm{SB}_F]$ generated by stably birational equivalence classes of integral schemes of finite type over $F$; for example $[\operatorname{Spec} F]_{sb}$ denotes the class of stably rational varieties over $F$.

Let $R$ be a (complete) discrete valuation ring whose residue field $k$ is algebraically closed of characteristic 0 and denote by $\bar{K}$ the algebraic closure of the fraction field of $R$. Nicaise–Shinder showed using the weak factorization theorem that there exists a unique ring homomorphism called stable birational volume

$$\mathrm{Vol}_{sb} \colon \mathbb{Z}[\mathrm{SB}_{\bar{K}}] \longrightarrow \mathbb{Z}[\mathrm{SB}_k]$$

such that for any proper strictly semi-stable $R$-scheme[2] $\mathcal{X}$ with geometric generic fibre $X_{\bar{\eta}}$ and special fibre $X$ with irreducible components $X_i$ for $i \in I$:

$$\mathrm{Vol}_{sb}(X_{\bar{\eta}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} \, [X_J]_{sb} \,, \quad \text{where } X_J := \bigcap_{j \in J} X_j \tag{1.3.1}$$

This immediately implies the following obstruction to (stable) rationality:

**Theorem 1.3.1** ([NS19])**.** *Let $R$ be a (complete) discrete valuation ring with algebraically closed residue field $k$ of characteristic $0$. Let $\mathcal{X}$ be a proper strictly semi-stable $R$-scheme with geometric generic fibre $X_{\bar{\eta}}$ and special fibre $X$. Denote the irreducible components of the special fibre by $X_i$ with $i \in I$. If*

$$\mathrm{Vol}_{sb}(X_{\bar{\eta}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} \, [X_J]_{sb} \neq [\operatorname{Spec} k]_{sb},$$

*then $X_{\bar{\eta}}$ is not stably rational.*

[2] A *strictly semi-stable $R$-scheme* $\mathcal{X}$ is a separated scheme flat and of finite type over $R$ such that the generic fibre is smooth and the special fibre is a geometrically reduced snc scheme whose irreducible components are Cartier divisors in $\mathcal{X}$, see Definition 2.3.2.

The method has been successfully applied using tropical geometry to rationality problems by Nicaise–Ottem [NO22], who showed that a very general quartic fivefold is not stably rational. The idea is to construct a strictly semi-stable family over the dvr $\mathbb{C}[[t]]$ whose generic fibre is a quartic fivefold and whose special fibre consists of two components $X_1$ and $X_2$ such that $X_1$ and $X_2$ are (stably) birational and the intersection $X_{12} \coloneqq X_1 \cap X_2$ is birational to a special quartic double fourfold; $X_{12}$ is known to be stably irrational by [HPT18a]. Then the stable birational volume formula (1.3.1) of that family reads

$$[X_1]_{sb} + [X_2]_{sb} - [X_{12}]_{sb} = 2[X_1]_{sb} - [X_{12}]_{sb} \neq [\operatorname{Spec}\mathbb{C}]_{sb} \in \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}].$$

Thus the geometric generic fibre is not stably rational by Theorem 1.3.1. Since stable rationality specializes in smooth proper families, the very general quartic fivefold is stably irrational.

Nicaise–Ottem construct such families via tropical geometry, which reduces the problem to finding suitable subdivision of polytopes. Moe [Moe23] studied this approach in his thesis and used it to show that very general hypersurfaces of degree $d \geq 4$ and dimension at most $(d+1)2^{d-4}$ over a field of characteristic 0 are stably irrational by reducing to [Sch19b].

Recently, the stable birational volume in (1.3.1) has been extended to detect $R$-equivalence classes by Hotchkiss–Stapleton [HS25]; in particular it can handle retract rationality. This also uses the weak factorization theorem; hence it is currently restricted to characteristic 0.

## 1.4. Chow-theoretic analogue and applications

Motivated by the result on quartic fivefolds in [NO22], Pavic–Schreieder [PS23] constructed a cycle-theoretic analogue of the motivic method and showed that the very general quartic fivefold admits no integral decomposition of the diagonal. To state the analogue, consider the following complex for a proper strictly semi-stable family $\mathcal{X}$ over a discrete valuation ring $R$:

$$\bigoplus_{j\in I} \mathrm{CH}_1(X_j) \xrightarrow{\Phi_{\mathcal{X}}} \bigoplus_{i\in I} \mathrm{CH}_0(X_i) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0, \tag{1.4.1}$$

where $X_i$ with $i \in I$ are the irreducible components of the special fibre, $\iota_i \colon X_i \hookrightarrow \mathcal{X}$ denote the natural inclusions, and $\Phi_{\mathcal{X}}$ is defined by $\Phi_{\mathcal{X}} \coloneqq \sum_{i,j} \iota_j^* \iota_{i*}$. If the discrete valuation ring $R$ is complete and the generic fibre of $\mathcal{X}$ admits a decomposition of the diagonal, then the complex (1.4.1) is exact by Grothendieck's version of Hensel's lemma [EGAIV, Theorem 18.5.17]. This observation led to an obstruction to a decomposition of the diagonal of the geometric generic fibre if the special fibre is a chain of Cartier divisors and the residue field of $R$ is algebraically closed, see [PS23, Theorem 1.2]. The key step is a detailed analysis of the complex (1.4.1) under finite extension of the discrete valuation ring $R'/R$ and suitable resolutions of the base-change $\mathcal{X} \times_R R'$. We extend this analysis to arbitrary configurations of the special fibre (Theorem 3.1.1).

Building on [PS23] and the analysis in Chapter 3, we provide in joint work with Schreieder a flexible obstruction to an integral decomposition of the diagonal.

**Theorem 1.4.1** (Theorem 4.3.8)**.** *Let $R$ be a discrete valuation ring with algebraically closed residue field $k$ of characteristic $p \geq 0$ and let $\Lambda$ be a torsion ring in which $p$ is invertible, if $p > 0$. Let $\mathcal{X}$ be a strictly semi-stable $R$-scheme such that the geometric generic fibre $X_{\bar{\eta}}$ is smooth and irreducible. Assume that the special fibre $X = X_0 \cup X_1$ consists of two irreducible components $X_0$ and $X_1$ which are isomorphic to open subschemes of affine space (of dimension at least 2) and whose intersection $X_{01} \coloneqq X_0 \cap X_1$ is an integral $k$-variety.*

*If $X_{01}$ admits no $\Lambda$-decomposition of the diagonal, then $X_{\bar{\eta}}$ admits no $\Lambda$-decomposition of the diagonal.*

A key advantage is that the $R$-scheme $\mathcal{X}$ does not need to be proper, which in practice avoids computing resolutions and allows to remove unnecessary irreducible components of the special fibre by working directly with a suitable open subscheme.

In the joint work with Schreieder, we use this method to extend the result of [Moe23] to retract irrationality and also to positive characteristic.

**Theorem 1.4.2** (Theorem 4.1.1)**.** *Let $k$ be a field of characteristic different from 2. Then a very general hypersurface $X \subset \mathbb{P}^{N+1}_k$ of degree $d \geq 4$ and dimension $N \leq (d+1)2^{d-4}$ does not admit a decomposition of the diagonal, hence is neither stably nor retract rational.*

We also obtain a weaker bound for the irrationality of hypersurface in characteristic 2, see Theorem 4.1.2. In subsequent joint work with Zhang, we extended the above bound to complete intersections, which improves earlier results of Nicaise–Ottem [NO22, Theorem 7.5].

**Theorem 1.4.3** (Theorem 5.1.2)**.** *Let $k$ be a field of characteristic different from 2. Then a very general complete intersection of multidegree $(d_1, \dots, d_s)$ and dimension*

*$N \geq 4$ does not admit a decomposition of the diagonal if some $d_i \geq 4$ and the Fano index*

$$N + s + 1 - d_1 - \cdots - d_s \leq (d_i + 1)2^{d_i - 4} - \left\lfloor \frac{d_i + 2}{2} \right\rfloor .$$

*In particular, such complete intersections are not retract rational.*

## 1.5. Zero-cycles over Laurent fields

The degree map

$$\deg\colon \mathrm{CH}_0(X) \longrightarrow \mathbb{Z}$$

is an isomorphism for a (smooth) projective rationally connected variety $X$ over an algebraically closed field. It is natural to consider other arithmetically interesting fields, e.g. local fields, fields of cohomological dimension 1. We focus here on the field of Laurent series $k((t))$ over an algebraically closed field $k$ of characteristic 0.

It follows from Graber–Harris–Starr [GHS03, Theorem 1.2] together with Greenberg's approximation theorem [Gre66] that a smooth projective rationally connected variety $X$ over $k((t))$ always contains a $k((t))$-rational point, in particular the degree map is surjective. The kernel of the degree map for $X$ is torsion by a pull-push argument; this uses that the degree map is an isomorphism over $\overline{k((t))}$.

In [EW16], Esnault–Wittenberg ask whether there exists a smooth proper rationally connected threefold $X$ over the Laurent field $\mathbb{C}((t))$ such that

$$\deg\colon \mathrm{CH}_0(X) \longrightarrow \mathbb{Z}$$

is not injective. Such examples over the local field $\mathbb{Q}_3$ were constructed by Parimala–Suresh [PS95, Proposition 6.2]. Tian used the minimal model program to show

**Theorem 1.5.1** ([Tia20, Theorem 1.1])**.** *Let $k$ be an algebraically closed field of characteristic 0 and let $X$ be a smooth projective rationally connected variety over the field $k((t))$. Assume that $\dim X \leq 3$ or that the Tate conjecture for one-cycles on surfaces holds, i.e. the $l$-adic cycle class map*

$$\mathrm{CH}_1(S) \otimes \mathbb{Q}_l \longrightarrow H^2_{\acute{e}t}(S, \mathbb{Q}_l(1))$$

*is surjective for any smooth projective surface $S$ over a finite field. Then the degree map induces an isomorphism $\mathrm{CH}_0(X) \simeq \mathbb{Z}$.*

The bijectivity result for the étale cycle class map of Saito-Sato [SS10], see also

Bloch's simplified proof in [EW16, Appendix], relates the injectivity of the degree map for a smooth projective variety $X$ over $k((t))$ with the exactness of the complex (1.4.1) for some strictly semi-stable model $\mathcal{X}$ of $X$ over $k[[t]]$ (if it exists), see the argument in the proof Corollary 3.5.6 for details. Thus, the analysis of the complex (1.4.1) in Chapter 3 yields the following result.

**Theorem 1.5.2** (Corollary 3.5.6)**.** *Let $X$ be a smooth, projective variety over a Laurent field $k((t))$ with $k$ algebraically closed of characteristic $0$. Assume that $X$ admits a strictly semi-stable projective model $\mathfrak{X} \to \operatorname{Spec} k[[t]]$. If $X$ admits geometrically a $\Lambda$-decomposition of the diagonal, then the degree map $\deg\colon \mathrm{CH}_0(X) \otimes \Lambda \to \Lambda$ is an isomorphism.*

The ideas presented in Chapter 6 may lead to an improved version of the result without the assumption on the existence of a strictly semi-stable model over $k[[t]]$.

## 1.6. Algebraic equivalence in strictly semi-stable families

Recall that an $l$-cycle $\alpha \in \mathrm{CH}_l(X)$ on an integral variety over a field $k$ is called algebraically equivalent to 0, if there exists a smooth integral variety $T$, an algebraic cycle $Z \in \mathrm{CH}_{l+\dim T}(T \times_k X)$ and two $k$-rational points $t_0, t_1 \in T$ such that $\alpha = Z_{t_0} - Z_{t_1} \in \mathrm{CH}_l(X)$, see [Ful98, Definition 10.3]. If $k$ is perfect, then $T$ can be assumed to be a smooth projective integral curve, see [ACMV17, Theorem 1]. Voisin asks in [Voi19] whether there exists a smooth projective rationally connected complex variety with a one-cycle which lies in the kernel of the cycle class map and is not algebraically equivalent to 0. In other words, whether the Griffiths group of one-cycles is non-trivial for some (smooth projective) rationally connected complex variety.

Such an example might be constructible by considering a suitable degeneration to an snc scheme as in Theorem 1.4.1 in order to reduce to a known example of a smooth projective (not rationally connected) variety with non-trivial Griffith group of one-cycles, see for instance [Sch92]. We provide a first step in this direction by showing that algebraic equivalence specializes for strictly semi-stable degenerations in equicharacteristic 0 in the following sense:

Let $\mathcal{X}$ be a proper strictly semi-stable scheme over a discrete valuation ring $R$ and denote its generic fibre by $X_\eta$. We denote by $X_i$ with $i \in I$ the irreducible components of the special fibre of $\mathcal{X}$. Consider the homomorphism

$$\mathrm{rsp}\colon \mathrm{CH}_l(X_\eta) \longrightarrow \bigoplus_{i\in I} \mathrm{CH}_l(X_i)/\operatorname{Im}\Phi_{\mathcal{X}}, \quad \alpha \mapsto (\iota_i^*\tilde{\alpha})_i,$$

where $\iota_i\colon X_i \hookrightarrow \mathcal{X}$ is the natural inclusion, $\Phi_{\mathcal{X}} := \sum_{i,j\in I} \iota_i^*\iota_{j*}$ similar to the definition in (1.4.1), and $\tilde{\alpha} \in \mathrm{CH}_{l+1}(\mathcal{X})$ is an $(l+1)$-cycle whose restriction to $X_\eta$ is $\alpha$. This is indeed a well-defined homomorphims by the localization exact sequence, see [Ful98, Proposition 1.8].

**Proposition 1.6.1** (Corollary 6.5.4)**.** *Let $R$ be a discrete valuation ring with algebraically closed residue field $k$ of characteristic $0$ such that $R$ is a localization of a finite type $k$-algebra. Let $\mathcal{X}$ be a strictly semi-stable $R$-scheme with generic fibre $X_\eta$ and special fibre $X$. Denote the irreducible components of $X$ by $X_i$ with $i \in I$. Then* rsp *induces a morphism on the Chow group of $l$-cycles modulo algebraic equivalence*

$$\mathrm{rsp}\colon A_l(X_\eta) \longrightarrow \bigoplus_{i\in I} A_l(X_i)/\operatorname{Im}\Phi_{\mathcal{X}}.$$

## 1.7. Declaration of own work in joint projects

The key idea to use a non-proper version of the map $\Phi_{\mathcal{X}}$ from (1.4.1) in the project [LS25] (Chapter 4) is due to Stefan Schreieder. We worked together on the precise statement of the main obstruction theorem Theorem 4.3.2 and I wrote down the argument for the theorem. I also worked out the section on the double cone construction Section 4.4 (using a construction of Moe) and the combinatorics in Section 4.6.

The collaboration [LZ25] (Chapter 5) started while Guoyun Zhang visited the Leibniz Universität Hannover (October 2024 - September 2025). I had the idea of using commutative algebra to approach the case of complete intersection. The precise framework (Section 5.2) grew out of several joint discussions. A few parts in the section on the base examples (Section 5.3) and the section on complete intersections of small Fano index (Section 5.4.2) are due to Guoyun Zhang. The remaining parts are written by myself.

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# 2. Preliminaries

## 2.1. Notations and conventions

For an abelian group $G$ and an integer $l \in \mathbb{Z}_{\geq 0}$, we denote by $G/l$ the tensor product $G \otimes_{\mathbb{Z}} \mathbb{Z}/l\mathbb{Z}$. All rings are assumed to be commutative with 1. The *characteristic of a ring* $\Lambda$ is the smallest positive integer $c \in \mathbb{Z}_{\geq 1}$ such that any element in $\Lambda$ is $c$-torsion; it is zero if no such integer exists. The *exponential characteristic* of a field $k$ is 1 if $k$ has characteristic zero and it is equal to the characteristic of $k$ otherwise. Unless stated otherwise, the generators of a polynomial ring are assumed to have degree 1.

Let $R$ be a ring. By an $R$-scheme we always mean a separated scheme of finite type over $R$. If $R = k$ is a field, then we also say *algebraic scheme over* $k$. A $k$*-variety* is an integral algebraic scheme over a field $k$.

For an $R$-scheme $X$ and an $R$-algebra $A$, we denote the fibre product by $X \times_R A \coloneqq X \times_{\operatorname{Spec} R} \operatorname{Spec} A$ or simply by $X_A$. We sometimes omit the ring $R$, if it is clear from the context.

## 2.2. Chow groups and Fulton's specialization map

Let $Y$ be a separated scheme of finite type over a regular Noetherian ring $R$; for our purposes $R = k$ is a field or a discrete valuation ring. The *group of* $l$*-cycles* of $Y$ is the free abelian group generated by subvarieties of dimension $l$. The *Chow group of* $l$*-cycles* $\mathrm{CH}_l(Y)$ of $Y$ is the quotient of the group of $l$-cycles by rational equivalence. For a ring $\Lambda$, we denote the tensor product $\mathrm{CH}_l(Y) \otimes_{\mathbb{Z}} \Lambda$ by $\mathrm{CH}_l(Y, \Lambda)$.

Let $R$ be a discrete valuation ring with fraction field $K$ and residue field $k$. Let $\mathcal{X} \to \operatorname{Spec} R$ be a flat $R$-scheme with generic fibre $X = \mathcal{X} \times_R K$ and special fibre $Y = \mathcal{X} \times_R k$. Then, for any ring $\Lambda$, there is a specialization map on Chow groups

$$\mathrm{sp} : \mathrm{CH}_i(X, \Lambda) \longrightarrow \mathrm{CH}_i(Y, \Lambda),$$

defined as follows. If $\Lambda = \mathbb{Z}$ and $z = [Z] \in Z_i(X)$ is represented by an $i$-dimensional subvariety $Z \subset X$, then $\mathrm{sp}(z)$ is represented by the restriction of the closure of $Z$ in $\mathcal{X}$ to $Y$. (This restriction could be empty, in which case $\mathrm{sp}(z) = [\emptyset] = 0$.) This extends $\mathbb{Z}$-linearly to a well-defined map by an argument of Fulton (see [Ful75, §4.4], [Ful98, §20.3], or [Sch21b, proof of Theorem 8.2]). The case of arbitrary coefficients follows from this by functoriality of the tensor product. If $k$ is algebraically closed, then the above map induces a well-defined map

$$\mathrm{sp}\colon \mathrm{CH}_i(\bar{X}, \Lambda) \longrightarrow \mathrm{CH}_i(Y, \Lambda),$$

where $\bar{X} = X \times_K \bar{K}$ denotes the base change to an algebraic closure. (In [Ful75, §4.4], [Ful98, §20.3], this is shown for the completion $\hat{R}$ of $R$; the above case then follows via precomposing with the natural map $\mathrm{CH}_i(\bar{X}) \to \mathrm{CH}_i(X \times_K \overline{\mathrm{Frac}(\hat{R})})$.)

We will need the following specific result on Fulton's specialization map.

**Lemma 2.2.1.** *Let $\Lambda$ be a ring and let $R$ be a discrete valuation ring with fraction field $K$ and residue field $k$. Let $p : \mathcal{X} \to \operatorname{Spec} R$ and $q : \mathcal{Y} \to \operatorname{Spec} R$ be flat $R$-schemes of finite type. Denote by $X_\eta, Y_\eta$ and $X_0, Y_0$ the generic and special fibres of $p$, $q$, respectively. Assume that $Y_\eta$ is geometrically integral and that there is a geometrically integral component $Y_0' \subset Y_0$, such that $A = \mathcal{O}_{\mathcal{Y}, Y_0'}$ is a discrete valuation ring and consider the flat $A$-scheme $\mathcal{X}_A \to \operatorname{Spec} A$, given by base change of $p$. Then Fulton's specialization map induces a specialization map*

$$\mathrm{sp}\colon \mathrm{CH}_i(X_\eta \times_K \bar{K}(Y_\eta), \Lambda) \longrightarrow \mathrm{CH}_i(X_0 \times_k \bar{k}(Y_0'), \Lambda),$$

*where $\bar{K}$ and $\bar{k}$ denote algebraic closures of $K$ and $k$, respectively, such that the following holds:*

*(1) $\mathrm{sp}$ commutes with pushforwards along proper maps and pullbacks along regular embeddings;*

*(2) If $\mathcal{X} = \mathcal{Y}$ and $X_0$ is integral, then $\mathrm{sp}(\delta_{X_\eta}) = \delta_{X_0}$, where $\delta_{X_\eta} \in \mathrm{CH}_0(X_\eta \times_K \bar{K}(X_\eta), \Lambda)$ and $\delta_{X_0} \in \mathrm{CH}_0(X_0 \times_k \bar{k}(X_0), \Lambda)$ denote the diagonal points.*

*Proof.* By functoriality of the tensor product, it suffices to prove the lemma in the case where $\Lambda = \mathbb{Z}$. The lemma is then stated under the assumption that $p$ and $q$ are proper with connected fibres in [PS23, Lemma 5.8], but the proof does not need those assumptions. □

## 2.3. Strictly semi-stable degenerations

**Definition 2.3.1.** Let $k$ be a field. An snc scheme of dimension $n$ over $k$ is a geometrically reduced algebraic scheme $Y$ over $k$ with irreducible components $Y_i$, $i \in I$, such that for any subset $J \subset I$, the (scheme-theoretic) intersection $Y_J := \bigcap_{j \in J} Y_j$ is smooth over $k$ and, if non-empty, equidimensional of dimension $n + 1 - |J|$.

We recall the definition of strictly semi-stable schemes over a discrete valuation ring, see e.g. [deJ96] and [Har01, Definition 1.1].

**Definition 2.3.2.** Let $R$ be a discrete valuation ring with fraction field $K$ and residue field $k$. A strictly semi-stable $R$-scheme $\mathcal{X} \to \operatorname{Spec} R$ is an irreducible, reduced, separated scheme which is flat and of finite type over $R$ with the following properties:

- the generic fibre $X = \mathcal{X} \times_R K$ is smooth over $K$;
- the special fibre $Y = \mathcal{X} \times_R k$ is an snc scheme over $k$;
- each component of the special fibre $Y$ is a Cartier divisor on $\mathcal{X}$.

Let $R$ be a discrete valuation ring and let $R \subset \tilde{R}$ be an unramified extension of discrete valuation rings, i.e. an extension $R \to \tilde{R}$ such that $m_{\tilde{R}} = m_R \tilde{R}$ where $m_R$ and $m_{\tilde{R}}$ are the maximal ideal of $R$ and $\tilde{R}$, respectively. Then for every strictly semi-stable $R$-scheme $\mathfrak{X}$, the base change $\mathfrak{X}_{\tilde{R}} = \mathfrak{X} \times_R \tilde{R}$ is a strictly semi-stable $\tilde{R}$-scheme, see e.g. [Har01, Proposition 1.3]. For finite ramified extension of dvr's the base-change $\mathfrak{X}_{\tilde{R}}$ becomes a strictly semi-stable $\tilde{R}$-scheme after a finite sequence of blow-ups.

**Proposition 2.3.3** ([Har01, Proposition 2.2])**.** *Let $R \subset \tilde{R}$ be a finite ramified extension of discrete valuation rings. Let $\mathfrak{X}$ be a strictly semi-stable $R$-scheme with special fibre $Y$. Assume that the irreducible components of $Y$ are geometrically integral. Then there exists a finite sequence of blow-ups $\tilde{\mathfrak{X}} := V^m \to V^{m-1} \to \cdots \to V^0 =: \mathfrak{X}_{\tilde{R}}$ such that $\tilde{\mathfrak{X}}$ is a strictly semi-stable $\tilde{R}$-scheme and the center of each blow-up $V^{i+1} \to V^i$ is an irreducible component of the special fibre of $V^i$.*

## 2.4. Very general elements of a family

We recall the notion of a very general element in a family, as stated for instance in [Sch25b, Section 2.5].

**Definition 2.4.1.** Let $k$ be a field and let $\pi\colon \mathcal{X} \to B$ be a proper flat morphism of algebraic $k$-schemes with $B$ geometrically integral. We say that a closed point $b \in B$ is *very general (with respect to $\pi$)* if there exist (algebraically closed) field extensions $K/\kappa(b)$ and $L/k(B)$ and an isomorphism $\varphi\colon K \to L$ of fields which induces an isomorphism

$$X_b \times_{\kappa(b)} K \simeq \mathcal{X} \times_{k(B)} L,$$

where $X_b := \pi^{-1}(b)$ denotes the fibre over $b$.

If $B$ is a fine moduli space or a parameter space of some class of algebraic varieties and $\pi\colon \mathcal{X} \to B$ is its universal family, we often simply say that $X_b$ is very general if $b \in B$ is.

**Remark 2.4.2.** If $k$ is an uncountable algebraically closed field, then [Via13, Lemma 2.1] shows that there exists a subset $U \subset B(k)$, which is the intersection of countably many non-empty open subsets of $B$, such that any point in $U$ is very general in the above sense.

We say an algebraic scheme $X$ over a field $L$ *degenerates* to an algebraic scheme $Y$ over an algebraically closed field $k$, if there exist a dvr $R$ with residue field $k$ and fraction field $K$, a flat proper morphism $\mathcal{X} \to \operatorname{Spec} R$, and an injection of fields $K \hookrightarrow L$ such that $Y \simeq \mathcal{X} \times_R k$ and $X \simeq \mathcal{X} \times_R L$, see for example [Sch19b, Section 2.6]. The following lemma is well-known.

**Lemma 2.4.3.** *Let $k$ be a field and let $\pi\colon \mathcal{X} \to B$ be a proper flat morphism of algebraic $k$-schemes such that $B$ is geometrically integral.*

*(i) If $\operatorname{trdeg}_{k_0} k \geq \dim B$, where $k_0 \subset k$ is a subfield over which $\pi$ is defined, then there exists a very general element $b \in B$.*

*(ii) Let $k'/k$ be a field extension. If $b \in B$ is very general with respect to $\pi$, then any closed point $b' \in B_{k'}$ lying over $b$ is very general with respect to $\pi_{k'}\colon X_{k'} \to B_{k'}$.*

*(iii) Up to a base-change, a very general fibre $X_b$ degenerates to the fibre $X_0$ over any closed point $0 \in B$ in the above sense.*

Note that there exists a subfield $k_0 \subset k$, which is finitely generated over its prime field, such that $\pi$ is defined over $k_0$, as $\pi\colon \mathcal{X} \to B$ is a morphism of algebraic schemes over $k$.

*Proof.* We start proving (i). Let $k_0 \subset k$ be a subfield such that $\pi$ is defined over $k_0$, i.e. there exists a morphism $\pi_0\colon \mathcal{X}_0 \longrightarrow B_0$ of algebraic schemes over $k_0$ such that all squares in the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"] \arrow[d] & B \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\ \mathcal{X}_0 \arrow[r, "\pi_0"] & B_0 \arrow[r] & \operatorname{Spec} k_0 \end{tikzcd}$$

are Cartesian. Assuming $\operatorname{trdeg}_{k_0} k \geq \dim B$, there exists a finite field extension $k^+/k$ and a homomorphisms of fields $k_0(B_0) \to k^+$. By the universal property of the fibre product, there exists a morphism $\iota\colon \operatorname{Spec} k^+ \to B$ of $k$-schemes such that the composition $\operatorname{Spec} k^+ \to B \to B_0$ factors through $\operatorname{Spec} k_0(B_0)$. In particular, $\iota$ determines a closed point $b \in B$ and we claim that $b$ is very general. Indeed, note that $\mathcal{X} = \mathcal{X}_0 \times_{B_0} B$ and the morphisms

$$\operatorname{Spec} k^+ \xrightarrow{\iota} B \longrightarrow B_0 \quad \text{and} \quad \operatorname{Spec} k(B) \xrightarrow{\eta} B \longrightarrow B_0,$$

factor through $\operatorname{Spec} k_0(B_0)$, where $\eta$ is the inclusion of the generic point. Thus

$$\begin{tikzcd} X_b \times_{\kappa(b)} k^+ \arrow[r] \arrow[d] & \mathcal{X}_0 \times_{B_0} \operatorname{Spec} k_0(B_0) \arrow[d] \\ \operatorname{Spec} k^+ \arrow[r] & \operatorname{Spec} k_0(B_0) \end{tikzcd} \qquad \begin{tikzcd} X_\eta \arrow[r] \arrow[d] & \mathcal{X}_0 \times_{B_0} \operatorname{Spec} k_0(B_0) \arrow[d] \\ \operatorname{Spec} k(B) \arrow[r] & \operatorname{Spec} k_0(B_0). \end{tikzcd}$$

are Cartesian squares. Here, $\kappa(b)$ denotes the residue field of the closed point $b$, which is a subfield of $k^+$ and $X_\eta$ denotes the generic fibre of the morphism $\pi\colon \mathcal{X} \to B$. In particular, it suffices to find (algebraically closed) field extensions $K/k^+$ and $L/k(B)$ together with an isomorphism of field $K \to L$ which fixes $k_0(B_0)$, see also the argument in [Via13, Lemma 2.1]. The existence of such fields $K$ and $L$ is clear, as any two algebraically closed field extension of the same transcendence degree over the same field are abstractly isomorphic.

By a similar argument as above, the statement (ii) reduces to the following question on fields. Given an isomorphism of algebraically closed fields $\varphi\colon K \to L$ with subfields $\kappa(b) \subset K$ and $k(B) \subset L$ as well as field extensions $\kappa(b')/\kappa(b)$ and $k'(B_{k'})/k(B)$, there exist algebraically closed fields $K'$ and $L'$ and an isomorphism $\varphi'\colon K' \to L'$ such that $K'$ contains $\kappa(b')$ and $K$ as subfields, $L'$ contains $k'(B_{k'})$ and $L$ as subfields, and $\varphi'$ extends $\varphi$. The existence follows by the same reasoning as in the proof of (i).

We turn to the proof of (iii). Let $0 \in B$ be any closed point. Up to replacing $k$ by an algebraically closed field extension whose transcendence degree over $k$ is at least $\dim B$, we can assume by (i) that there exists a closed point $b \in B$, which is very general in the sense of Definition 2.4.1. It clearly suffices to show that the fibre $X_b := \pi^{-1}(b)$ degenerates to $X_0$. By a Bertini-type argument, there exists a geometrically integral curve $C \subset B$ containing 0 and $b$, see e.g. [CP16, Corollary 1.9]. Up to replacing $C$ by its normalization, we can assume that there exists a smooth geometrically integral curve $C$ and a flat proper morphism $\pi_C \colon \mathcal{X}_C \to C$ and two closed points $c_0, c \in C$ such that $c \in C$ is very general in the sense of Definition 2.4.1 and the fibres over $c_0$ and $c$ are isomorphic to $X_0$ and $X_b$, respectively. Thus we see that $X_b$ degenerates to $X_0$, as $\mathcal{O}_{C,c_0}$ is a dvr. □

**Example 2.4.4.** We will use the following two examples of families over a field $k$.

(a) Complete intersections in $\mathbb{P}^N$: Let $\mathbf{d} := (d_1, \ldots, d_s) \in \mathbb{Z}^s_{\geq 1}$ be a collection of positive integers, where $1 \leq s < N$ is an integer. Then the affine scheme

$$B_1 := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d_1)) \times_k \cdots \times_k H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d_s))$$

parametrizes sets of $s$ polynomials $f_1, \ldots, f_s \in k[x_0, \ldots, x_N]$ with $\deg f_i = d_i$. Let $\mathcal{X}_1 \subset B_1 \times_k \mathbb{P}^N$ be the closed subscheme such that the fibre of $\mathcal{X}_1 \to B_1$ over a point corresponding to $f_1, \ldots, f_s$ is the closed subscheme of $\mathbb{P}^N$ cut out by the homogeneous ideal $(f_1, \ldots, f_s)$. There exists an open subscheme $B \subset B_1$ such that the fibre over any point $b \in B$ has dimension $N - s$. Thus the morphism $\mathcal{X} := \mathcal{X}_1 \times_{B_1} B \to B$ is a family of complete intersections of multidegree $\mathbf{d}$.

Other choices for $B$ include (an open subscheme of) a product of Grassmannians or an iteratated Grassmannian bundle over $\operatorname{Spec} k$. We refer the reader to [Ben12, §2.2] and [DL23, §1.2] for more details on moduli spaces for complete intersections.

(b) The parameter space of hypersurfaces of multidegree $\mathbf{d} := (d_1, \ldots, d_s)$ in the product of projective spaces $\mathbb{P}^{M_1}_k \times_k \cdots \times_k \mathbb{P}^{M_s}_k$ is given by

$$B := \mathbb{P}\left(H^0\left(\mathbb{P}^{M_1}, \mathcal{O}_{\mathbb{P}^{M_1}}(d_1)\right) \otimes_k \cdots \otimes_k H^0\left(\mathbb{P}^{M_s}, \mathcal{O}_{\mathbb{P}^{M_s}}(d_s)\right)\right).$$

Consider the universal family $\mathcal{X} \to B$ of multidegree $\mathbf{d}$ hypersurfaces in the product of projective spaces $\mathbb{P}^{M_1}_k \times_k \cdots \times_k \mathbb{P}^{M_s}_k$.

**Remark 2.4.5.** Let $\mathcal{X} \to B := \mathbb{P}(H^0(\mathbb{P}^N_k, \mathcal{O}(d)))$ be the universal family of degree $d$ hypersurfaces in $\mathbb{P}^N$. Its geometric generic fibre is given by

$$\mathcal{X}_{\bar{\eta}} = \left\{ \sum_{I \in \mathbb{Z}^{N+1}_{\geq 0} : |I| = d} t_I x^I = 0 \right\} \subset \mathbb{P}^N_{k(B)},$$

where $|I| := i_0 + \cdots + i_N$ and $t_I \in k(B)$ denotes the element corresponding to the monomial $t^I = t^{i_0} \cdots t^{i_N}$ in $H^0\left(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)\right)$ for each $I = (i_0, \ldots, i_N) \in \mathbb{Z}^{N+1}_{\geq 0}$.

Hence, we see that a hypersurface is very general in the sense of Definition 2.4.1 if and only (up to an isomorphism) the coefficients of a defining equation are algebraically independent over the prime field of $k$.

## 2.5. Basic constructions in toric geometry

Throughout this section, let $k$ be a field. A (split) toric variety is a normal $k$-variety $X$ which contains a torus $T = (k^*)^{\dim X}$ as an open dense subset. Toric varieties are also constructed from collections of cones, called fans, and many properties of the toric variety relate to combinatorical data of the fan. We recall the construction of toric varieties from fans, introduce some notation and state a few standard facts from toric geometry.

Let $N \simeq \mathbb{Z}^n$ be a lattice and denote by $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$ its dual lattice. Consider the natural duality pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. Morover, we denote the tensor products $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, respectively.

A *(strongly convex, rational, polyhedral or short: scrp) cone* in $N_{\mathbb{R}}$ is a set

$$\sigma := \mathrm{Cone}(S) := \left\{ \sum_{u \in S} \lambda_u u : \lambda_u \geq 0 \right\} \subset N_{\mathbb{R}},$$

where $S \subset N$ is a finite subset such that $\sigma \cap (-\sigma) = \{0\}$. The elements in $S$ are called *generators* of $\sigma$. The *dimension* of $\sigma$ is the dimension of its linear span in $N_{\mathbb{R}}$. The *dual cone* of $\sigma$ is the subset

$$\sigma^{\vee} := \{ m \in M_{\mathbb{R}} : \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \} \subset M_{\mathbb{R}},$$

which is again a (rational, polyhedral, not necessarily strongly convex) cone. Gordan's lemma says that $\sigma^{\vee} \cap M$ is a finitely generated semi-group such that $k[\sigma^{\vee} \cap M]$ is an integral domain, see e.g. [CLS11, Proposition 1.2.17]. Then the *affine toric* *variety* $U_\sigma$ associated to a cone $\sigma$ is $U_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$.

A *face* $\tau$ of an (scrp) cone $\sigma$ is a subset of the form

$$\tau = \{u \in \sigma : \langle m, u\rangle = 0\} \subset \sigma$$

for some $m \in \sigma^\vee$ and it is again an (scrp) cone. A collection of (scrp) cones $\Sigma$ is called a *fan* if each face of a cone is also in $\Sigma$ and if the intersection of two cones in $\Sigma$ is a common face of both. These two conditions ensure that we can glue the affine toric varieties associated to each cone $\sigma \in \Sigma$ along the intersection of the cones to obtain a toric variety, which we denote by $X_\Sigma$. We denote the set of $n$-dimensional cones in the fan $\Sigma$ by $\Sigma(n)$ and we usually call 1-dimensional cones *rays*. Recall also the following facts from toric geometry:

(i) A $\mathbb{Z}$-linear map of lattices $N \to N'$ is *compatible* with a pair of fans $\Sigma$ in $N_\mathbb{R}$ and $\Sigma'$ in $N'_\mathbb{R}$ if the $\mathbb{R}$-linear extension maps every cone $\sigma \in \Sigma$ into a cone $\sigma' \in \Sigma'$. Such a compatible map of lattices gives rise to a toric morphism of the associated toric varieties, i.e. a morphism of the varieties such that the restriction to the tori is a group homomorphism, see [Oda78, Theorem 4.1].

(ii) The orbit-cone correspondence yields an inclusion-reversing bijection between cones in a fan and orbits of the torus action of its associated toric variety, see e.g. [Oda78, Theorem 4.2]. If $\sigma \in \Sigma$ is a cone in the fan, then we denote the corresponding orbit in $X_\Sigma$ by $O(\sigma)$ and its Zariski closure by $V(\sigma)$.

(iii) The generators $v_i$ of the 1-dimensional faces $\rho_i$ of an (scrp) cone $\sigma \subset N_\mathbb{R}$ yield a set of generators for $\sigma$. Up to scaling $v_i$, we can assume that $v_i \in (N \setminus \{0\}) \cap \rho_i$ is of minimal length. Then the $v_i$'s are called the *minimal generators* of $\sigma$.

(iv) We say that a cone is *simplicial* if its minimal generators are $\mathbb{R}$-linear independent. A cone is called *regular* if its minimal generators form part of a $\mathbb{Z}$-basis of the lattice $N$. We say a fan is *simplicial* or *regular*, if all its cones are so. Note that regularity of the fan is equivalent to the regularity of the associated toric variety, see [Oda78, Theorem 4.3].

For a more detailed account of the constructions and the facts in toric geometry over arbitrary fields, we refer the reader to [Oda78] and the survey [Dan78]. An earlier description of toric varieties over algebraically closed fields can be found in [KKMS]. We refer the reader to [CLS11] for a modern treatment of toric geometry.

## 2.6. Monomial orderings and Gröbner basis

We recall a few basic notions related to monomial ordering of polynomial rings following the reference [CLO25, Section 2.2].

Let $k$ be a field and let $R = k[x_1, \dots, x_n]$ be the polynomial ring over $k$ in $n$ variables. A *monomial ordering* on $R$ is a well-ordering[1] on the set of monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ such that for every $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$

$$x^\alpha > x^\beta \implies x^{\alpha+\gamma} > x^{\beta+\gamma},$$

see [CLO25, Section 2.2 Definition 1]. A monomial ordering $>$ is called *graded* if $|\alpha| > |\beta|$ implies $x^\alpha > x^\beta$ for every $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, where $|\alpha| \coloneqq \alpha_1 + \cdots + \alpha_n$. The most important example for Chapter 5 is the graded lexicographical ordering.

**Example 2.6.1** ([CLO25, Section 2.2 Definition 5])**.** Consider the relation $>_{\text{grlex}}$ on the set of monomials $x^\alpha$, $\alpha \in \mathbb{Z}^n_{\geq 0}$, on $R = k[x_1, \dots, x_n]$ given by $x^\alpha >_{\text{grlex}} x^\beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the left-most non-zero entry in $\alpha - \beta \in \mathbb{Z}^n$ is positive. Then $>_{\text{grlex}}$ is a graded monomial ordering on $R$, called the *graded lexicographic ordering*.

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field $k$ and fix a monomial ordering on $R$. For a polynomial $f = \sum_\alpha a_\alpha x^\alpha \in R$ we define the *leading monomial* of $f$ as

$$\mathrm{LM}(f) \coloneqq \max\{x^\alpha : a_\alpha \neq 0\}, \tag{2.6.1}$$

where the maximum is taken with respect to the choosen monomial ordering. We say that the leading monomials of a collection $f_1, \dots, f_r$ of polynomials in $R$ are *relatively prime* if for all $i, j \in \{1, \dots, r\}$ with $i \neq j$ we have

$$\gcd(\mathrm{LM}(f_i), \mathrm{LM}(f_j)) = 1.$$

**Example 2.6.2.** Consider the polynomials

$$f = 3x^2y^3 + 6x^3y^2 - 5xy + 5, \quad g = x^5 + x^2y^2 + y^3 + xy - 1, \quad h = y^2 + y + 1 \in k[x, y].$$

The leading monomials with respect to the graded lexicographic ordering $(x > y)$ are

$$\mathrm{LM}(f) = x^3y^2, \quad \mathrm{LM}(g) = x^5, \quad \mathrm{LM}(h) = y^2.$$

[1] a total ordering such that any non-empty subset admits a smallest element

In particular, we see that the leading monomials of $g$ and $h$ are relatively prime, while the leading monomials of $f$ and $g$ are not.

**Definition 2.6.3.** Let $f \in k[x_1, ..., x_n]$ be a polynomial of degree $d$. Then

$$f^h(x_0, \dots, x_n) \coloneqq x_0^d f\left(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right)$$

is a homogeneous polynomial of degree $d$ in $k[x_0, ..., x_n]$, called the *homogenization* of $f$. Let $I \subset k[x_1, ..., x_n]$ be an ideal. Then we define the *homogenization* of I to be the ideal

$$I^h = \left(f^h : f \in I\right) \subset k[x_0, x_1, \dots, x_n],$$

which is a homogeneous ideal in $k[x_0, x_1, ..., x_n]$.

A collection of polynomials whose leading monomials are relatively prime in the above sense behave well under homogenization.

**Proposition 2.6.4.** *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field $k$ and fix a graded monomial ordering on $R$. If the leading monomials of a collection of polynomials $f_1, \dots, f_r \in R$ of degree at least 1 are relatively prime, then $I^h = (f_1^h, \dots, f_r^h)$. In particular, the scheme $\{f_1^h = \cdots = f_r^h = 0\} \subset \mathbb{P}^n_k$ is the projective closure of $\{f_1 = \cdots = f_r = 0\} \subset \mathbb{A}^n_k$, if $k$ is algebraically closed.*

*Proof.* This is a consequence of the theory of Gröbner basis. The assumption on the relative primeness of the leading monomials implies that $f_1, \dots, f_r$ is a Gröbner basis for $I$ by [CLO25, Section 2.9, Proposition 4 and Theorem 3]. The homogenization of a Gröbner basis of $I$ is a basis of $I^h$, see e.g. [CLO25, Section 8.4, Theorem 4]. The claim about the projective closure is shown for example in [CLO25, Section 8.4, Theorem 8]. $\square$

# 3. Tian's conjecture

## 3.1. Introduction

Kontsevich–Tschinkel [KT19] and Nicaise–Shinder [NS19] study the behaviour of rationality and stable rationality in families in characteristic 0. In particular for degenerations of a smooth projective variety, they construct a motivic obstruction to (stable) rationality which depends only on the special fibre of the degeneration. This approach was successfully applied by Nicaise–Ottem [NO22] to show the stable irrationality of quartic fivefolds and several complete intersections by reducing to previously known irrationality results [HPT18a, Sch19b]. Building on [Sch19b], [Moe23] uses the approach in [NO22] to improve Schreieder's logarithmic bound for stably irrational hypersurfaces in characteristic 0.

Motivated by the cycle-theoretic approaches to stable rationality in [Voi15, CTP16, Sch19a], Pavic–Schreieder [PS23] introduce a Chow-theoretic analogue of the motivic approach for strictly semi-stable schemes $\mathfrak{X}$ over a dvr $R$. These are integral separated regular flat $R$-schemes such that the generic fibre $X = \mathfrak{X}_\eta$ is a smooth variety and the special fibre is a geometrically reduced simple normal crossing divisor $Y = \bigcup_{i \in I} Y_i$ with irreducible components $Y_i$. For each strictly semi-stable and proper $\mathfrak{X} \to \operatorname{Spec} R$ consider the complex

$$\bigoplus_{j \in I} \mathrm{CH}_1(Y_j) \xrightarrow{\Phi_{\mathfrak{X}}} \bigoplus_{i \in I} \mathrm{CH}_0(Y_i) \xrightarrow{\sum_i \deg} \mathbb{Z} \longrightarrow 0, \tag{3.1.1}$$

where $\Phi_{\mathfrak{X}} = \sum\limits_{i \in I} \sum\limits_{j \in I} \iota_i^* \iota_{j*}$ with $\iota_i \colon Y_i \hookrightarrow \mathfrak{X}$ the natural inclusion for $i \in I$.

**Theorem 3.1.1.** *Let $R$ be a discrete valuation ring with algebraically closed residue field $k$, and let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable projective $R$-scheme. Assume that the geometric generic fibre admits a decomposition of the diagonal (e.g. is retract rational). Then the complex (3.1.1) is exact after base-change to any field extension $L/k$, i.e. the complex (3.1.1) is exact for the strictly semi-stable family $\mathfrak{X} \times_R A \to \operatorname{Spec} A$, where $A/R$ is any unramified extension of dvr's with induced extension $L/k$ of residue fields.*

The theorem implies that the complex (3.1.1) is exact modulo $m$ for all $m \in \mathbb{N}$, as the tensor product is right exact, see also Corollary 3.5.4 for a more general version. While the residue field of $A$ does not need to be algebraically closed, we would like to emphasize that the assumption on the residue field of $R$ being algebraically closed is crucial, see Remark 3.5.5. Theorem 3.1.1 improves [PS23, Theorem 1.2 (2)] which shows the exactness of the complex tensored with $\mathbb{Z}/2$ if the special fibre is a chain of Cartier divisors. We can remove the restriction to $\mathbb{Z}/2$-coefficients and allow an arbitrary configuration of the special fibre in the strictly semi-stable family. The theorem of Pavic–Schreieder implies the (retract) irrationality of very general quartic fivefolds [PS23, Theorem 1.1] and of very general complete intersections of two cubics in $\mathbb{P}^7$ [LS23, Theorem 1.1]. One might hope to approach the other complete intersections in [NO22] and the hypersurfaces in [Moe23] with Theorem 3.1.1.

As a consequence of Theorem 3.1.1, we study in this chapter the Chow group of zero-cycles of geometrically retract rational varieties over Laurent fields $k((t))$ where $k$ is algebraically closed. Colliot-Thélène shows in [CT83, Theorem A (iv)] that the Chow group of degree 0 zero-cycles is trivial for geometrically rational surfaces over fields of characteristic 0 and cohomological dimension at most 1. In particular [CT83] applies to Laurent fields in characteristic 0. More generally, Tian considers rationally connected varieties over Laurent fields in characteristic 0 and shows the triviality of the Chow group of zero-cycles unconditionally in dimension at most 3 and in all dimensions if the Tate conjecture for surfaces holds, see [Tia20, Theorem 1.1]. We obtain a partial result in this direction and a similar result in positive characteristic.

**Corollary 3.1.2.** *Let $X$ be a smooth, projective variety over a Laurent field $k((t))$ with $k$ algebraically closed. Assume that $X$ admits a strictly semi-stable projective model $\mathfrak{X} \to \operatorname{Spec} k[[t]]$. If $X$ admits geometrically an integral decomposition of the diagonal, e.g. $X$ is geometrically retract rational, then*

*(i) the degree map* $\deg\colon \operatorname{CH}_0(X) \to \mathbb{Z}$ *is an isomorphism if* $\operatorname{char} k = 0$*;*

*(ii) the degree map* $\deg\colon \operatorname{CH}_0(X)/l \to \mathbb{Z}/l$ *is an isomorphism for every $l$ coprime to* $p = \operatorname{char} k$*, if $p > 0$.*

For smooth families $\mathfrak{X} \to \operatorname{Spec} k[[t]]$, Corollary 3.1.2 (i) follows from [Kol04, Theorem 2 (2)]. Over $\mathbb{C}((t))$, the result is known for rationally simply connected varieties by [Pir11, Theorem 1.5]. Note that there are smooth projective varieties over $\mathbb{C}$ admitting an integral decomposition of diagonal which are not rationally connected and thus in particular not rationally simply connected, e.g. Barlow surfaces, see [ACTP17, Proposition 1.9] and [Voi17, Corollary 2.2].

**Remark 3.1.3.** We prove a more general version of Corollary 3.1.2 for a $\Lambda$-decomposition of the diagonal (see (1.2.1)) and over fraction fields of excellent, henselian discrete valuation rings with algebraically closed residue field $k$ in Corollary 3.5.6. The latter corollary is applicable to varieties, which are geometrically Enriques surfaces ($\Lambda = \mathbb{Z}[1/2]$) or geometrically Godeaux surfaces ($\Lambda = \mathbb{Z}[1/5]$ for a general Godeaux surface).

A variant of the homomorphism $\Phi$ in (3.1.1) is used to determine the kernel of the degree map $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ for certain geometrically rational surfaces over local fields in [Dal05a, Dal05b] and for a cubic threefold over $\mathbb{C}((u))((t))$ in [Mad08].

We briefly sketch the strategy for the two results: The assumption of Theorem 3.1.1 ensures that the generic fibre of $\mathfrak{X} \to \operatorname{Spec} R$ admits a decomposition of the diagonal after a finite field extension. This implies by [PS23, Theorem 1.2 (1)] that the complex (3.1.1) is exact for a suitable resolution $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ after a finite base change $\tilde{R}/R$. From this we aim to deduce the exactness over $R$. The issue is that the suitable resolution $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ has many more components in the special fibre, so the associated complexes (3.1.1) are quite different. We describe the suitable resolution $\tilde{\mathfrak{X}}$ étale locally by toric geometry and obtain a global description via simplicial complexes. This way we can reduce Theorem 3.1.1 to a combinatorial problem which we solve in Proposition 3.4.9.

Corollary 3.1.2 follows from Theorem 3.1.1 via Fulton's localization exact sequence

$$\mathrm{CH}_1(Y) \longrightarrow \mathrm{CH}_1(\mathfrak{X}) \longrightarrow \mathrm{CH}_0(X) \longrightarrow 0$$

by using Saito-Sato's [SS10] bijectivity result for the étale cycle class map.

Section 3.2 contains some toric intersection theory and a description of a particular affine toric singularity and its resolution. The main technical aspects of this chapter are contained in Section 3.3, where we describe the irreducible components of the resolution in terms of a subdivision of the dual complex of snc schemes. In Section 3.4, we use the results from Section 3.3 to compare the complex (3.1.1) with the complex of a suitable resolution after a finite base change by constructing two auxiliary functions. The main result Theorem 3.1.1 and Corollary 3.1.2 are explained in Section 3.5. In Section 3.6, we provide two concrete examples, for which we illustrate the key proposition - Proposition 3.4.9 - by direct computations. Throughout this chapter, we use the following setup and refer to it whenever needed.

**Setup 3.1.4.** Let $R$ be a discrete valuation ring with residue field $k$. Let $\tilde{R}/R$ be a finite extension of discrete valuation rings of ramification index $r$ with induced extension $L/k$ of residue fields. Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable $R$-scheme with special fibre $Y = \mathfrak{X} \times_R k$. Assume that the irreducible components of $Y$ are geometrically integral. Let $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ be a resolution of $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$ as in Proposition 2.3.3. We denote the natural morphism $\tilde{\mathfrak{X}} \to \mathfrak{X}$ as well as its restriction to the special fibre $\tilde{Y} = \tilde{\mathfrak{X}} \times_{\tilde{R}} L$ by $q$.

## 3.2. Resolution of a toric singularity

The proof of Theorem 3.1.1 relies on the analysis of a certain type of toric singularity together with an iterative description of its resolution as well as a combinatorial description of the intersection of toric curves with toric divisors. We recall the necessary intersection theory from [CLS11] in the first part of this section. The second part consists of a description of the toric singularity and a resolution process.

### 3.2.1. Some intersection theory

We recall the required intersection theory which is described by combinatorical data of the fan. Here, we are interested in the intersection of toric divisors with one-cycles, which is described in [CLS11, Section 6.4].

Let $\Sigma$ be a simplicial fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$. Recall from Section 2.5 that we denote by $\Sigma(m)$ the set of $m$-dimensional cones in $\Sigma$ and rays are one-dimensional cones $\rho \in \Sigma(1)$. Let $\tau \in \Sigma(n-1)$ be the intersection of two cones $\sigma, \sigma'$ in $\Sigma(n)$. Since $\Sigma$ is simplicial, we can label the minimal generators $u_i$ of $\sigma, \sigma'$, and $\tau$ such that

$$\sigma = \operatorname{Cone}(u_1, \dots, u_n), \quad \sigma' = \operatorname{Cone}(u_2, \dots, u_{n+1}), \quad \tau = \operatorname{Cone}(u_2, \dots, u_n). \tag{3.2.1}$$

Moreover, since the generators $u_1, \dots, u_{n+1}$ are linearly dependent, they satisfies a (up to multiplication with a constant unique) linear relation, called the *wall relation*

$$\sum_{i=1}^{n+1} b_i u_i = 0. \tag{3.2.2}$$

Since $u_2, \dots, u_n$ are linearly independent, we see that $u_1, u_{n+1}$ are both non-zero.

**Proposition 3.2.1** ([CLS11, Proposition 6.4.4])**.** *Let $\tau = \sigma \cap \sigma' \in \Sigma(n-1)$ be as in (3.2.1) and let $V(\tau)$ be the Zariski closure of the orbit corresponding to $\tau$.*

*Moreover, let* $\rho_i = \mathbb{R}_{\geq 0} u_i$ *be the rays in* $N_{\mathbb{R}}$ *generated by* $u_i$*. Then for every ray* $\rho \in \Sigma(1)$ *with associated divisor* $D_\rho = V(\rho)$ *in the toric variety* $X_\Sigma$*, the intersection number* $D_\rho \cdot V(\tau)$ *is given by*

$$D_\rho \cdot V(\tau) = \begin{cases} \frac{\operatorname{mult}(\tau)}{\operatorname{mult}(\sigma)} & \text{if } \rho = \rho_1, \\ \frac{\operatorname{mult}(\tau)}{\operatorname{mult}(\sigma')} = \frac{b_{n+1} \operatorname{mult}(\tau)}{b_1 \operatorname{mult}(\sigma)} & \text{if } \rho = \rho_{n+1}, \\ \frac{b_i \operatorname{mult}(\tau)}{b_1 \operatorname{mult}(\sigma)} & \text{if } \rho = \rho_i, i \neq 1, n+1, \\ 0 & \text{otherwise,} \end{cases}$$

*where the multiplicity* $\operatorname{mult}(\gamma)$ *of a cone* $\gamma \subset N_{\mathbb{R}}$ *with minimal generators* $v_1, \ldots, v_k$ *is the index of the sublattice* $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k \subset N_\gamma = (\mathbb{R}v_1 + \cdots + \mathbb{R}v_k) \cap N$*.*

*Sketch of the proof.* Using the uniqueness up to scalar of the wall relation together with the observation that $\sum_{\rho \in \Sigma(1)} (D_\rho \cdot V(\tau))\, u_\rho = 0$ in $N_{\mathbb{R}}$, where $u_\rho$ is the minimal generator of the ray $\rho \in \Sigma(1)$, it suffices to compute $D_{\rho_1} \cdot V(\tau)$ which can be done explicitly, see e.g. [CLS11, Lemma 6.4.2]. □

**Remark 3.2.2.** If the fan $\Sigma$ is regular, the multiplicity of all cones is 1. Moreover, we can assume without loss of generality that $b_1 = 1$ and thus also $b_{n+1} = 1$ by the above proposition. Hence the formula in Proposition 3.2.1 reduces to

$$D_\rho \cdot V(\tau) = \begin{cases} 0 & \rho \notin \{\rho_1, \ldots, \rho_{n+1}\}, \\ 1 & \rho = \rho_1,\ \rho_{n+1}, \\ b_i & \rho = \rho_i, \quad i \neq 1, n+1. \end{cases} \tag{3.2.3}$$

### 3.2.2. A particular toric singularity

We discuss the toric description of a particular singularity. This type of singularity appears naturally in our problem: We consider a strictly semi-stable $R$-scheme $\mathfrak{X} \to \operatorname{Spec} R$ for some discrete valuation ring $R$. Étale locally at a point of the special fibre the scheme $\mathfrak{X}$ looks like

$$\{t - x_1 \cdots x_n = 0\} \subset \mathbb{A}^n_R, \tag{3.2.4}$$

where $t \in m_R \subset R$ is a uniformizer and $n \geq 1$. Let $\tilde{R}/R$ be a finite ramified extension of dvr's. The base-change $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$ admits a resolution to a strictly semi-stable $\tilde{R}$-scheme by multiple blow-ups of the irreducible components, see Proposition 2.3.3.

During the blow-ups the local equation looks like (3.2.5) below, see [Har01, proof of Proposition 2.2]. Thus the following lemma is a toroidal description of the behaviour under these blow-ups.

**Lemma 3.2.3.** *For any $m, n \geq 1$ and $r_1, \dots, r_m \geq 1$, the affine variety*

$$Z = \{x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m} - y_0 y_1 \cdots y_n = 0\} \subset \mathbb{A}_k^{n+m+1} \tag{3.2.5}$$

*is the affine toric variety corresponding to the fan spanned by the cone*

$$\sigma = \sigma_{n,r_1,\dots,r_m} = \operatorname{Cone}\left(\{e_i + r_i f_j : i = 1, \dots, m,\ j = 0, 1, \dots, n\}\right) \tag{3.2.6}$$

*in $N_{\mathbb{R}} = \mathbb{R}^{m+n}$, where $e_1, \dots, e_m, f_1, \dots, f_n$ is a basis of $N = \mathbb{Z}^m \oplus \mathbb{Z}^n$ and we additionally set $f_0 = 0 \in N$. Moreover, we have*

*(a) for each $i = 1, \dots, m$ and $j = 0, 1, \dots, n$ the ray $\rho_{i,j} = \mathbb{R}_{\geq 0}(e_i + r_i f_j)$ correspond to the irreducible subvariety $D_{\rho_{i,j}} = V(x_i, y_j)$.*

*(b) for $i = 1, \dots, m$ the natural projection $\overline{\pi} = \mathrm{pr}_i \oplus 0 \colon \mathbb{Z}^m \oplus \mathbb{Z}^n \to \mathbb{Z}$ onto the $i$-th coordinate is compatible with the fan given by $\sigma$ and $\mathbb{R}_{\geq 0} \cdot 1 \subset N'_{\mathbb{R}} = \mathbb{R}$ and the corresponding toric morphism is the projection onto the $i$-th coordinate $x_i$: $\pi = \mathrm{pr}_i \colon Z \to \mathbb{A}_k^1$.*

*(c) for $r_1 \geq 1$ define $e_{m+1} \coloneqq e_1 + f_1$ and $r_{m+1} \coloneqq r_1 - 1$; then the blow-up of $Z$ along $V(x_1, y_1)$ is given by the fan spanned by the two cones*

$$\sigma' = \operatorname{Cone}(\{e_i : i = 1, \dots, m+1\} \cup \{e_i + r_i f_j : i = 1, \dots, m+1,\ j = 2, \dots, n\})$$
$$\sigma'' = \operatorname{Cone}(\{e_i : i = 2, \dots, m+1\} \cup \{e_i + r_i f_j : i = 2, \dots, m+1,\ j = 1, \dots, n\}).$$

*Proof.* Let $\sigma$ be the cone as defined in (3.2.6). We claim that the dual cone $\sigma^\vee$ is given by

$$\sigma^\vee = \operatorname{Cone}(e_1^*, \dots, e_m^*, f_1^*, \dots, f_n^*, r_1 e_1^* + \cdots + r_m e_m^* - f_1^* - \cdots - f_n^*).$$

If this claim is true, the first statement follows immediately, because

$$k[\sigma^\vee \cap M] = {}^{k[x_1, \dots, x_m, y_0, \dots, y_n]}\!/\!_{(x_1^{r_1} \cdots x_m^{r_m} - y_0 y_1 \cdots y_n)}.$$

To prove the claim, note that "$\supseteq$" is obvious. We prove the other inclusion: Let

$$\sum_{i=1}^{m} a_i e_i^* + \sum_{j=1}^{n} b_j f_j^* \in \sigma^\vee$$

for some $a_i, b_j \in \mathbb{R}$. By definition, the coefficients $a_i, b_j$ satisfy

$$a_i \geq 0, \quad a_i + r_i b_j \geq 0, \text{ for } i = 1, \dots, m \text{ and } j = 1, \dots, n.$$

Let $-\lambda = \min\{b_1, \dots, b_n, 0\} \leq 0$, then

$$\sum_{i=1}^{m} a_i e_i^* + \sum_{j=1}^{n} b_j f_j^* = \underbrace{\lambda}_{\geq 0}(r_1 e_1^* + \dots + r_m e_m^* - f_1^* - \dots - f_n^*) + \sum_{i=1}^{m} \underbrace{(a_i - \lambda r_i)}_{a_i + r_i b_j \text{ or } a_i \geq 0} e_i^* + \sum_{j=1}^{n} \underbrace{(\lambda + b_j)}_{\geq 0} f_j^*.$$

This shows "$\subseteq$" and thus the claim. We turn to the proof of item (a): Note that the orthogonal part of the cone $\rho_{i,j}$ for $i = 1, \dots, m$ and $j = 0, \dots, n$ is given by

$$\rho_{i,j}^\perp = \begin{cases} \operatorname{Cone}\left(e_1^*, \dots, \widehat{e_i^*}, \dots, e_m^*, f_1^*, \dots, f_n^*\right) & \text{if } j = 0, \\ \operatorname{Cone}\left(e_1^*, \dots, \widehat{e_i^*}, \dots, e_m^*, f_1^*, \dots, \widehat{f_j^*}, \dots, f_n^*, r_1 e_1^* + \dots r_m e_m^* - f_1^* - \dots - f_n^*\right) & \text{if } j \neq 0. \end{cases}$$

Hence, the distinguished point of $\rho_{i,j}$ is the point $(x_1, \dots, x_m, y_0, y_1, \dots, y_n) \in \mathbb{A}_k^{n+m+1}$ with $x_a = \delta_{a,i}$ and $y_b = \delta_{b,j}$ where $\delta_{\cdot,\cdot}$ is the Kronecker delta. Thus statement (a) follows from the orbit-cone correspondence. Statement (b) follows directly from the construction. For statement (c), note that the blow-up of (3.2.5) along $V(x_1, y_1)$ is given by

$$\begin{aligned} x_1 &= x_1' y_1, \quad (x_1')^{r_1} x_2^{r_2} \cdots x_m^{r_m} y_1^{r_1 - 1} - y_0 y_2 \cdots y_n = 0, \\ y_1 &= y_1' x_1, \quad (x_1)^{r_1 - 1} x_2^{r_2} \cdots x_m^{r_m} - y_0 y_1' y_2 \cdots y_n = 0. \end{aligned}$$

Thus we find that the corresponding dual vectors are given by

$$\begin{aligned} &(e_1')^* = e_1^* - f_1^*,\ (e_i')^* = e_i^*,\ (e_{m+1}')^* = f_1^*,\ (f_j')^* = f_j^*, \quad i = 2, \dots, m,\ j = 2, \dots, n, \\ &(f_1'')^* = f_1^* - e_1^*,\ (e_i'')^* = e_i^*,\ (f_j'')^* = f_j^*, \quad i = 1, \dots, m,\ j = 2, \dots, n, \end{aligned}$$

Hence, we see that

$$\begin{aligned} e_{m+1}' = e_1 + f_1,\ e_i' = e_i,\ f_j' = f_j, \quad i = 1, \dots, m,\ j = 2, \dots, n, \\ e_1'' = e_1 + f_1,\ e_i'' = e_i,\ f_j'' = f_j, \quad i = 2, \dots, m,\ j = 1, \dots, n. \end{aligned}$$

Thus the description for $\sigma'$ and $\sigma''$ follows from the description of $Z$ in (3.2.5). $\square$

**Remark 3.2.4.** For $m = 2$ and $r_1 = r_2 = 1$, a similar computation is done in [Shi22, Lemma 2.2].

**Remark 3.2.5.** Up to relabeling the coordinates the blow-up description in (c) also holds for the blow-up along $V(x_i, y_j)$ where we think of $f_0$ as $f_0 = 0$ and do the calculation formally. More precisely, the blow-up of $V(x_1, y_0)$ corresponds to the subdivision of $\sigma$ into the two cones

$$\begin{aligned}\sigma' &= \operatorname{Cone}\left(\{e_i + r_i f_j : i = 1, \ldots, m+1,\ j = 1, \ldots, n\}\right), \\ \sigma'' &= \operatorname{Cone}\left(\{e_2, \ldots, e_{m+1}\} \cup \{e_i + r_i f_j : i = 2, \ldots, m+1,\ j = 1, \ldots, n\}\right),\end{aligned}$$

where $e_{m+1} = e_1$ and $r_{m+1} = r_1 - 1$.

In concrete examples one can work with an explicit resolution. For our purposes it suffices to work with some resolution and use its properties. The first such property is the following.

**Proposition 3.2.6.** *Let $\sigma = \sigma_{n,r}$ be a cone as in Lemma 3.2.3 with $m = 1$. Then any sequence of subdivisions of the same form as in Lemma 3.2.3(c) terminates with a regular fan $\tilde{\Sigma}$. Moreover, every minimal generator of a ray in $\tilde{\Sigma}$ lies in the hyperplane $\{e_1 = 1\}$ and is a lattice point of the lattice generated by $e_1, f_1, f_2, \ldots, f_n$ as defined in Lemma 3.2.3.*

**Remark 3.2.7.** Note that the subdivision from Lemma 3.2.3(c) corresponds to the blow-up of the irreducible components of the special fibre which is the procedure in Proposition 2.3.3.

*Proof.* The termination of this process is precisely [Har01, Proposition 2.2] which yields a resolution after some finite ramified base change, see also Proposition 2.3.3. Since the total space $\tilde{\mathfrak{X}}$ of a strictly semi-stable model is regular, the local model of the toric variety is regular. Alternatively, the explicit local descriptions and the termination condition of the algorithm in [Har01, Proposition 2.2] imply also the regularity of the refined fan.

The last statement follows from Lemma 3.2.3(c) by induction. Indeed, starting with $\sigma = \sigma_{n,r}$ for some $n, r \in \mathbb{N}$ we see that the minimal generators of each ray is of the form $e_1$ or $e_1 + r f_j$ for some $j = 1, \ldots, n$, i.e. they lie in the hyperplane $\{e_1 = 1\}$ and are lattice points. In the $j$-th step, we add a new vector $e_{j+1} \coloneqq e_{j'} + f_{k'}$ for some $j' = 1, \ldots, j$. $k' = 1, \ldots, n$. The new minimal generators after the blow-up are of the form

$$e_{j+1} + r_{j+1} f_k$$

for some $k = 1, \dots, n$ and $r_{j+1} \in \mathbb{Z}_{\geq 0}$. Thus all minimal generators lie in the hyperplane $\{e_1 = 1\}$ and are lattice points by induction. $\square$

## 3.3. Chow group of the resolution

We recall the statement of Theorem 3.1.1 from the introduction and the strategy laid out in the introduction. Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable scheme over a dvr $R$ with algebraically closed residue field. Assume that the geometric generic fibre of $\mathfrak{X} \to \operatorname{Spec} R$ admits a decomposition of the diagonal. Then the complex

$$\bigoplus_{j \in I} \mathrm{CH}_1\left(Y_{j,L'}\right) \xrightarrow{\Phi_{\mathfrak{X}_A}} \mathrm{CH}_0\left(Y_{i,L'}\right) \xrightarrow{\sum_i \deg} \mathbb{Z} \longrightarrow 0 \tag{3.3.1}$$

is exact for every unramified extension $A/R$ with induced extension $L'/k$ of residue fields. Recall that $\Phi_{\mathfrak{X}_A} = \sum_{i \in I} \sum_{j \in I} \iota_i^* \iota_{j*}$ where $\iota_i \colon Y_{i,L'} \hookrightarrow \mathfrak{X}_A$ is the natural inclusion. The assumption on the geometric generic fibre implies the exactness of (3.3.1) for a strictly semi-stable family $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ which is obtained by a finite (possibly ramified) base change and a resolution as in Proposition 2.3.3. We want to relate $\Phi_{\tilde{\mathfrak{X}}}$ with $\Phi_{\mathfrak{X}}$ such that we can deduce the exactness of the complex (3.3.1) for $\mathfrak{X}$ from the exactness of the complex (3.3.1) for $\tilde{\mathfrak{X}}$, which then proves Theorem 3.1.1. In this section, we describe the special fibre $\tilde{Y}$ of $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ using subdivisions of the dual complex of an snc scheme and express its Chow group of one-cycles in terms of cycles supported $Y$. Before that we recall dual complexes and subdivisions and introduce some necessary notation. The description of $\mathrm{CH}_1(\tilde{Y})$ is then used in Section 3.4 to relate $\Phi_{\mathfrak{X}}$ and $\Phi_{\tilde{\mathfrak{X}}}$. We follow [Hat02, Appendix] and [dFKX17, Definition 7] for the definition of $\Delta$-complexes.

**Definition 3.3.1** ($\Delta$-complex)**.** The notion of a $\Delta$-complex is defined by induction on the dimension. A 0-*skeleton* $\mathcal{C}^0$ is a collection of points, called *vertices*. Inductively, define an $n$-*skeleton* $\mathcal{C}^n$ by attaching a collection $\{\sigma_i : i \in I_n\}$ of simplices of dimension $n$ to an $(n-1)$-skeleton $\mathcal{C}^{n-1}$ via *characteristic maps* $\varphi_i \colon \partial\sigma_i \to \mathcal{C}^{n-1}$ and identifying points on the boundary $\partial\sigma_i$ of $\sigma_i$ with their image in $\mathcal{C}^{n-1}$. A (regular finite-dimensional unordered) $\Delta$-*complex* $\mathcal{C}$ is an $n$-skeleton $\mathcal{C}^n$ for some $n \in \mathbb{Z}_{\geq 1}$ such that all characteristic maps are embeddings; $n$ is called the *dimension* of $\mathcal{C}$.

Let $\mathcal{C}$ be a $\Delta$-complex. A simplex $\sigma$ of $\mathcal{C}$ of dimension $n$ is called an $n$-*simplex*; the set of $n$-simplices of $\mathcal{C}$ is denoted by $\mathcal{C}(n)$. A (proper) *face* of $\sigma$ is a simplex $\sigma'$ of $\mathcal{C}$ such that $\sigma' \subset \partial\sigma$.

Recall from graph theory: A vertex $v'$ in a graph is adjacent to a vertex $v$ if there exists an edge with endpoints $v$ and $v'$. We extend this definition to $\Delta$-complexes by considering the 1-skeleton of the $\Delta$-complex which is a graph.

**Definition 3.3.2** (Adjacent vertices)**.** Let $\mathcal{C}$ be a $\Delta$-complex and let $v \in \mathcal{C}(0)$ be a vertex. A vertex $v' \in \mathcal{C}(0)$ is called *adjacent to* $v$ if there exists an 1-simplex $\sigma$ such that $\partial\sigma = \{v, v'\}$. We denote the set of adjacent vertices of $v$ in $\mathcal{C}$ by $A_{\mathcal{C}}(v)$.

Moreover, we define walls of a $\Delta$-complex inspired by the definition of walls for toric varieties, see e.g. [Dan78, Section 10.5].

**Definition 3.3.3** (Walls)**.** Let $\mathcal{C}$ be a $\Delta$-complex of dimension $n$. A simplex $\tau \in \mathcal{C}(n-1)$ is a *wall* of $\mathcal{C}$ if $\tau$ is a common face of two simplices $\sigma_1, \sigma_2 \in \mathcal{C}(n)$. Equivalently if $\tau$ is the (set-theoretic) intersection of the simplices $\sigma_1, \sigma_2$.

We recall the definition of a subdivision, see e.g. [Mun84, §15].

**Definition 3.3.4** (Subdivision of $\Delta$-complex)**.** Let $\mathcal{C}$ be a $\Delta$-complex. A *subdivision* of $\mathcal{C}$ is a $\Delta$-complex $\mathcal{C}'$ such that

(i) every simplex $\sigma'$ of $\mathcal{C}'$ is contained in a simplex of $\mathcal{C}$.

(ii) every simplex $\sigma$ of $\mathcal{C}$ is a non-empty union of finitely many simplices of $\mathcal{C}'$.

We formally denote the subdivision by a map $\psi\colon \mathcal{C}' \to \mathcal{C}$. For a simplex $\sigma'$ of $\mathcal{C}'$, we denote the smallest simplex in $\mathcal{C}$ containing the simplex $\sigma'$ by $\psi_*\sigma'$. The non-empty union of simplices of $\mathcal{C}'$ in (ii) is denoted by $\psi^*\sigma$ for a simplex $\sigma$ of $\mathcal{C}$.

**Remark 3.3.5.** (i) It follows from item (ii) in Definition 3.3.4 that $\psi^*v$ consist of a single vertex for a vertex $v \in \mathcal{C}(0)$.

(ii) Note that $\psi^*\sigma$ is a subcomplex of $\mathcal{C}'$ for every simplex $\sigma$ of $\mathcal{C}$, i.e. a subset of $\mathcal{C}'$ which becomes a complex by restricting the characteristic maps of $\mathcal{C}'$ to that subset.

For such a subdivision we define relative versions of adjacent vertices and walls which become useful later on in Section 3.4.

**Definition 3.3.6.** Let $\mathcal{C}'$ be a subdivision of a $\Delta$-complex $\mathcal{C}$ in the above sense.

(i) For any vertex $v' \in \mathcal{C}'(0)$ we define the *set of relative adjacent vertices* as

$$A_\psi(v') := \begin{cases} A_{\mathcal{C}}(\psi_* v') & \text{if } \psi_* v' \in \mathcal{C}(0), \\ \emptyset & \text{otherwise,} \end{cases}$$

where $A_{\mathcal{C}}$ is defined in Definition 3.3.2.

(ii) A *relative wall* $\tau$ of $\mathcal{C}'$ is a wall of the subcomplex $\psi^*\sigma$ for some simplex $\sigma$ of $\mathcal{C}$. We denote the set of relative walls of the subdivision $\mathcal{C}'$ by $\mathrm{RelWall}(\psi)$.

(iii) Let $v' \in \mathcal{C}'$ be a vertex. We define the *set of lines through $v'$ with adjacent vertex in $\mathcal{C}$* as

$$\mathcal{L}_\psi(v') \coloneqq \left\{(\sigma, v) \in \mathcal{C}(1) \times \mathcal{C}(0) : v, v' \in \sigma \text{ and } \psi^* v \in A_{\mathcal{C}'}(v')\right\}.$$

In other words, $\mathcal{L}_\psi(v')$ is the set of 1-simplices $\sigma$ in $\mathcal{C}$ which contain $v'$ (viewed as an element in the $\Delta$-complex $\mathcal{C}$) such that one of the faces of $\sigma$ is adjacent to $v'$.

We recall the well-known definition of the dual complex of an snc-scheme, see e.g. [dFKX17, Definition 8].

**Definition 3.3.7** (Dual complex)**.** Let $Y = \bigcup_{i\in I} Y_i$ be a pure dimensional scheme with irreducible components $Y_i$ such that for every non-empty $J \subset I$

$$Y_J \coloneqq \bigcap_{i\in J} Y_j$$

is either empty or smooth and equidimensional of codimension $|J| - 1$. The *dual complex $\mathcal{D}(Y)$ of $Y$* is the (regular finite-dimensional unordered) $\Delta$-complex defined as follows. The vertices are the irreducible components of $Y$. For each non-empty $J \subset I$, we associate a $(|J|-1)$-simplex $\sigma_Z$ to each irreducible component $Z \subset Y_J$ with the obvious characteristic maps. We denote the irreducible subscheme associated to a simplex $\sigma$ of $\mathcal{D}(Y)$ by $Y_\sigma$ and call them the *strata* of $Y$.

**Remark 3.3.8.** Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable $R$-scheme for some discrete valuation ring $R$ with residue field $k$, see Definition 2.3.2. The special fibre $Y = \mathfrak{X} \times_R k$ is an snc scheme. Thus we obtain a dual complex $\mathcal{D}(Y)$ of $Y$.

**Lemma 3.3.9.** *Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable $R$-scheme with special fibre $Y$ and let $\tilde{R}/R$ be a finite ramified extension of dvr's with ramification index $r$. Let $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ be a resolution of the base-change $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$ as in Proposition 2.3.3. We denote the special fibre of $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ by $\tilde{Y}$. Then the dual complex $\mathcal{D}(\tilde{Y})$ of $\tilde{Y}$ is a subdivision of the dual complex $\mathcal{D}(Y)$ of $Y$, which we denote formally by a map $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$.*

The proof of Lemma 3.3.9 uses the following observation which allows us to translate between the étale local toric picture and the global picture via the dual complex.

**Remark 3.3.10.** Let $\sigma_Z \in \mathcal{D}(Y)$ be a simplex associated to a stratum $Z$ of $Y$. Then $Z$ is étale locally the (affine) toric variety associated to the cone $\sigma_{n,1}$ from Lemma 3.2.3 with $m = 1$, where $n = \dim \sigma_Z$. Moreover, Lemma 3.2.3(a) and the orbit-cone correspondence imply that this identification respects the strata of $Y$ in the sense that a stratum in the dual complex is mapped to the same stratum in the étale local toric picture.

*Proof of Lemma 3.3.9.* Let $\sigma_Z \in \mathcal{D}(Y)$ be a simplex associated to a stratum $Z$ of $Y$ and let $\sigma_{n,1}$ be the corresponding cone under the correspondence in Remark 3.3.10. Then the base-change $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$ corresponds étale locally to the cone $\sigma_{n,r}$ from Lemma 3.2.3 with the obvious morphism of lattices, namely scaling each generator of $N$ by $\frac{1}{r}$ except the first one. (Note that this morphism is compatible with the cones $\sigma_{n,r}$ and $\sigma_{n,1}$). The resolution $\tilde{\mathfrak{X}} \to \mathfrak{X}$ corresponds étale locally by Proposition 3.2.6 to a subdivision of the cone $\sigma_{n,r}$. Moreover, this subdivision of the cone is induced by a subdivision of the $n$-simplex and gives a simplicial fan, see Proposition 3.2.6. Since this construction is the étale local analogue of $\tilde{\mathfrak{X}} \to \mathfrak{X}$ and the correspondence in Remark 3.3.10 respects the strata of $Y$ and $\tilde{Y}$, we see that the constructed subdivision is a subdivision in the sense of Definition 3.3.4 and is equal to $\mathcal{D}(\tilde{Y})$, as claimed. □

**Remark 3.3.11.** Note that the correspondence between the étale local toric picture and the global picture via dual complexes does not hold in each step of the sequence of subdivisions in Proposition 3.2.6 as the cones obtained during this sequence of subdivision are in general not simplicial. Hence, we use the (global) description with dual complexes only for the strictly semi-stable models $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ and $\mathfrak{X} \to \operatorname{Spec} R$.

We describe the new components appearing in the resolution process of Proposition 2.3.3, which we described étale locally in Section 3.2.2. This enables us to explicitly describe the Chow group of one-cycles of the special fibres of the resolution, see Proposition 3.3.14.

**Proposition 3.3.12.** *With the same notation as in Setup 3.1.4, let $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$ be the subdivision of dual complexes from Lemma 3.3.9. Recall that each component in $Y$ and $\tilde{Y}$ corresponds to a vertex in $\mathcal{D}(Y)$ and $\mathcal{D}(\tilde{Y})$, respectively. For every vertex $\tilde{v} \in \mathcal{D}(\tilde{Y})(0)$, the irreducible component $\tilde{Y}_{\tilde{v}}$ is obtained by a finite sequence of blow-ups in smooth centers and Zariski locally trivial $\mathbb{P}^1$-bundles of the base-change $Y_{\psi_*\tilde{v},L}$ where $Y_{\psi_*\tilde{v}}$ is the irreducible subscheme of $Y$ associated to the simplex $\psi_*\tilde{v}$, see Definition 3.3.7.*

*Proof.* This follows from the description of the resolution process in [Har01, Proposition 2.2] together with the construction of the dual complex $\mathcal{D}(Y)$ and the toric construction of its subdivision. Indeed, Hartl's resolution process is iterative and in each step every component will be blown-up in a smooth (possibly empty) center and the new components in that step are étale locally trivial $\mathbb{P}^1$-bundles over the intersection of two components in the previous step, see Lemma 3.2.3 or [Har01, Proof of Proposition 2.2]. Since we blow-up in that step one of the two components, we see that the étale locally trivial $\mathbb{P}^1$-bundle admits a section. Thus it is a Zariski-locally trivial $\mathbb{P}^1$-bundle over the intersection of two components in the previous step. Hence each component $\tilde{Y}_{\tilde{v}}$ for $\tilde{v} \in \mathcal{D}(\tilde{Y})(0)$ is of the form described in the statement. $\square$

**Definition 3.3.13.** Let the notation be as in Proposition 3.3.12. Let $\tau \in \mathrm{RelWall}(\psi)$ be a relative wall as defined in Definition 3.3.6(ii). Let $\psi_*\tau$ be the smallest simplex of $\mathcal{D}(Y)$ containing $\tau$. Then $\tau$ corresponds via the toric description to a (Zariski) locally trivial $\mathbb{P}^1$-bundle $P_\tau$ over $Y_{\psi_*\tau,L}$. We denote the projection by $\tilde{q}_\tau \colon P_\tau \to Y_{\psi(\tau),L}$.

Additionally, we fix for any $\tau \in \mathrm{RelWall}(\psi)$ a vertex $v(\tau) \in \mathcal{D}(\tilde{Y})(0)$ such that $v(\tau)$ is a face of $\tau$. This implies via the toric description that the projective bundle $P_\tau$ lies inside $\tilde{Y}_{v(\tau)}$ and we denote the natural inclusion by $\tilde{\iota}'_\tau \colon \tilde{Y}_{v(\tau)} \hookrightarrow \tilde{Y}$.

**Proposition 3.3.14.** *Let the notation be as in Proposition 3.3.12 and Definition 3.3.13. Then the following hold:*

*(a) The Chow group $\mathrm{CH}_1(Y, \Lambda)$ of one-cycles on $Y$ is generated by cycles of the form*

$$\gamma = \sum_{v \in \mathcal{D}(Y)(0)} \iota'_{v\,*}\gamma_v \in \mathrm{CH}_1(Y, \Lambda), \tag{3.3.2}$$

*where $\gamma_v \in \mathrm{CH}_1(Y_v, \Lambda)$ is a one-cycle and $\iota'_v \colon Y_v \hookrightarrow Y$ is the natural inclusion.*

*(b) The Chow group $\mathrm{CH}_1(\tilde{Y}, \Lambda)$ of one-cycles on $\tilde{Y}$ is generated by cycles of the form*

$$\tilde{\gamma} = \sum_{v \in \mathcal{D}(Y)(0)} \tilde{\iota}'_{v\,*}\tilde{\gamma}_v + \sum_{\tau \in \mathrm{RelWall}(\psi)} \tilde{\iota}'_{\tau\,*}\tilde{q}_\tau^*\tilde{\alpha}_\tau \in \mathrm{CH}_1(\tilde{Y}, \Lambda), \tag{3.3.3}$$

*where $\tilde{\gamma}_v \in \mathrm{CH}_1(Y_{v,L}, \Lambda)$ is a one-cycle, $\tilde{\alpha}_\tau \in \mathrm{CH}_0\left(Y_{\psi_*\tau,L}, \Lambda\right)$ is a zero-cycle, and $\tilde{\iota}'_v \colon \tilde{Y}_v \hookrightarrow \tilde{Y}$, $\tilde{\iota}'_\tau \colon \tilde{Y}_{v(\tau)} \hookrightarrow \tilde{Y}$, and $\tilde{q}_\tau \colon P_\tau \to Y_{\psi(\tau),L}$ are the natural morphisms, see also Definition 3.3.13.*

**Remark 3.3.15.** (a) Part (b) says that the Chow group of one-cycles on $\tilde{Y}$ is governed by $\mathrm{CH}_1(Y, \Lambda)$ and by a toric part. Indeed, we can see (3.3.3) as

$$\tilde{\gamma} = \underbrace{\sum_{v \in \mathcal{D}(Y)(0)} \tilde{\iota}'_{v\,*} \tilde{\gamma}_v}_{\text{from } \mathrm{CH}_1(Y,\Lambda)} + \underbrace{\sum_{\tau \in \mathrm{RelWall}(\psi)} \tilde{\iota}'_{\tau\,*} \tilde{q}^*_\tau \tilde{\alpha}_\tau}_{\text{“toric part”}}.$$

Morally speaking, the difference between (3.3.2) and (3.3.3) lies in this toric part.

(b) Two examples of such a description can be found in Section 3.6.

*Proof.* Since any prime one-cycle, i.e. every irreducible, 1-dimensional subvariety, is contained in an irreducible component, statement (a) follows immediately. The same argument shows that $\mathrm{CH}_1(\tilde{Y}, \Lambda)$ is generated by one-cycles on the components of $\tilde{Y}$. We show statement (b) by using the structure of the components from Proposition 3.3.12: Recall the following two standard facts about Chow groups, see e.g. [Ful98, Theorem 3.3 (b) and Proposition 6.7 (e)]:

(i) Let $P = \mathbb{P}(E) \to W$ be the projectivization of a vector bundle $E$ on a smooth variety $W$ of relative dimension $\geq 2$. Then

$$\mathrm{CH}_1(P, \Lambda) \simeq \mathrm{CH}_0(W, \Lambda) \oplus \mathrm{CH}_1(W, \Lambda), \qquad \mathrm{CH}_0(P, \Lambda) \simeq \mathrm{CH}_0(W, \Lambda),$$

where the isomorphisms are given by pulling-back a cycle on $W$ along the flat map $P \to W$ and intersecting with a suitable power of the canonical line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ which is dual to the pull-back of the vector bundle $E$ on $W$ to $P = \mathbb{P}(E)$.

(ii) Let $\tilde{W}$ be the blow-up of a smooth variety $W$ along a smooth subvariety $Z$. Then

$$\mathrm{CH}_1(\tilde{W}, \Lambda) \simeq \mathrm{CH}_0(Z, \Lambda) \oplus \mathrm{CH}_1(W, \Lambda),$$

where the isomorphism is given by applying the isomorphism from (i) to the projective bundle $\mathbb{P}(\mathcal{N}_{Z/W}) \to Z$ and pushing the one-cycle forward to $\tilde{W}$ as well as pushing a one-cycle on $W \setminus Z$ forward along the natural morphism $W \setminus Z \to \tilde{W}$.

Let $\tilde{v} \in \mathcal{D}(\tilde{Y})(0)$ be a vertex. Assume first $\tilde{v} = \psi^* v$ for a vertex $v \in \mathcal{D}(Y)(0)$. (Note that $\psi^* v$ consists of a single vertex by Remark 3.3.5(i).) By Proposition 3.3.12, the component $\tilde{Y}_{\tilde{v}}$ of $\tilde{Y}$ is a sequence of smooth blow-ups of $Y_{v,L}$, i.e.

$$\tilde{Y}_{\tilde{v}} = \mathrm{Bl}_{Z_r} \mathrm{Bl}_{Z_{r-1}} \cdots \mathrm{Bl}_{Z_1} Y_{v,L}$$

where $Z_1, \ldots, Z_r$ are the smooth centers of the blow-ups. Then by fact (ii) we find

$$\mathrm{CH}_1(\tilde{Y}_{\tilde{v}}, \Lambda) \simeq \bigoplus_{i=1}^{r} \mathrm{CH}_0(Z_i, \Lambda) \oplus \mathrm{CH}_1\left(Y_{v,L}, \Lambda\right). \tag{3.3.4}$$

The $\mathrm{CH}_0(Z_i, \Lambda)$ parts correspond under the orbit-cone correspondence to some relative walls. Indeed, every zero-cycle is a (formal) linear combination of closed points. For each closed point the image under the isomorphism (3.3.4) is the one-cycle given by pulling back the point to the corresponding $\mathbb{P}^1$-bundle over $Z_i$ and pushing the one-cycle forward to $\tilde{Y}_{\tilde{v}}$ via the natural inclusion. This one-cycle is an irreducible rational curve which corresponds via the orbit-cone correspondence to a relative wall. Hence, we see that $\mathrm{CH}_1(\tilde{Y}_{\tilde{v}}, \Lambda)$ is described by one-cycles corresponding to relative walls and $\mathrm{CH}_1\left(Y_{v,L}, \Lambda\right)$.

We consider now the case $\tilde{v} \notin \psi^*\mathcal{D}(Y)(0)$, i.e. the simplex $\sigma = \psi_*\tilde{v}$ of $\mathcal{D}(Y)$ has $\dim(\sigma) \geq 1$. By Proposition 3.3.12, $\tilde{Y}_{\tilde{v}}$ is a sequence of smooth blow-ups and projective bundles over some $Y_{\sigma,L}$, i.e.

$$\tilde{Y}_{\tilde{v}} = \mathrm{Bl}_{Z_r} \mathrm{Bl}_{Z_{r-1}} \cdots \mathrm{Bl}_{Z_1} P_{\tilde{v}}$$

where $Z_1, \ldots, Z_r$ are the smooth centers of blow-ups and $P_{\tilde{v}}$ is a Zariski locally trivial $\mathbb{P}^1$-bundle over $W_{\tilde{v}}$ such that the intersection with a different component $\tilde{Y}_{\tilde{v}'}$ yields a section. Then the facts (i) and (ii) above imply

$$\mathrm{CH}_1(\tilde{Y}_{\tilde{v}}, \Lambda) \simeq \bigoplus_{i=1}^{r} \mathrm{CH}_0(Z_i, \Lambda) \oplus \mathrm{CH}_0(W_{\tilde{v}}, \Lambda) \oplus \mathrm{CH}_1(W_{\tilde{v}}, \Lambda).$$

The latter part $\mathrm{CH}_1(W_{\tilde{v}}, \Lambda)$ is contained in $\mathrm{CH}_1(Y_{\tilde{v}'}, \Lambda)$ and by the same argument as in the case $\tilde{v} \in \psi^*\mathcal{D}(Y)(0)$ we find that the $\mathrm{CH}_0(Z_i, \Lambda)$ parts and the $\mathrm{CH}_0(W_{\tilde{v}}, \Lambda)$ part correspond via the orbit-cone correspondence to some relative walls (in the above sense). Hence, we see that $\mathrm{CH}_1(\tilde{Y}_{\tilde{v}}, \Lambda)$ is described by one-cycles corresponding to relative walls and the $\mathrm{CH}_1$ which appear in a previous step of Proposition 2.3.3. Inductively we get that the $\mathrm{CH}_1$ of the "new" components (corresponding to vertices in $\mathcal{D}(\tilde{Y})(0) \backslash \psi^*\mathcal{D}(Y)(0)$) is described by one-cycles corresponding to the relative walls and $\mathrm{CH}_1(Y_{v,L}, \Lambda)$ for $v \in \mathcal{D}(Y)$. The one-cycles corresponding to the relative walls and the one-cycles supported on $Y_{v,L}$ might appear in multiple components, but they need to be counted once, as for an irreducible subvariety $Z$ of $\tilde{Y}_v$ and $\tilde{Y}_w$ the diagram with the natural inclusions

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & \tilde{Y}_v \arrow[d, hook, "\iota'_v"] \\ \tilde{Y}_w \arrow[r, hook, "\iota'_w"] & \tilde{Y}, \end{tikzcd}$$

commutes. This proves part (b) of the proposition. □

## 3.4. Analysis of the base-change

We recall our setup (Setup 3.1.4), which we fix throughout this entire section. Let $R$ be a discrete valuation ring with residue field $k$ (not necessarily algebraically closed) and let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable $R$-scheme with special fibre $Y$ whose irreducible components $Y_i$ (with $i \in I$) are geometrically integral. Let $\tilde{R}/R$ be a finite extension of dvr's with induced extension $L/k$ of residue fields. Let $\tilde{\mathfrak{X}}$ be a resolution of the base change $\mathfrak{X}_{\tilde{R}}$ from Proposition 2.3.3, i.e. $\tilde{\mathfrak{X}}$ a strictly semi-stable $\tilde{R}$-scheme; denote its special fibre by $\tilde{Y}$. Let $q\colon \tilde{\mathfrak{X}} \to \mathfrak{X}$ denote the natural morphism and also the restriction to the special fibre $q\colon \tilde{Y} \to Y$. Recall that the dual complex $\mathcal{D}(\tilde{Y})$ of $\tilde{Y}$ is a subdivision of the dual complex $\mathcal{D}(Y)$ of $Y$ which we denote formally by $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$, see Lemma 3.3.9. Consider the homomorphism

$$\bigoplus_{j\in I} \mathrm{CH}_1(Y_j,\Lambda) \xrightarrow{\Phi^\Lambda_{\mathfrak{X}}} \bigoplus_{i\in I} \mathrm{CH}_0(Y_i,\Lambda), \tag{3.4.1}$$

where $\iota_i\colon Y_i \hookrightarrow \mathfrak{X}$ is the natural inclusion and $\Phi^\Lambda_{\mathfrak{X}} = \sum\limits_{i\in I}\sum\limits_{j\in I} \iota_i^*\iota_{j*}$. In this section we relate (3.4.1) with $\Phi^\Lambda_{\tilde{\mathfrak{X}}}$. This relays on the following two auxiliary functions.

**Construction 3.4.1.** We construct a function

$$d\colon \mathcal{D}(\tilde{Y})(0) \times \mathcal{D}(Y)(0) \longrightarrow \mathbb{Z}_{\geq 0}, \tag{3.4.2}$$

which measures a "distance" between the vertices of $\mathcal{D}(\tilde{Y})$ to the vertices of $\mathcal{D}(Y)$.

Let $v' \in \mathcal{D}(\tilde{Y})(0)$ be a vertex and let $\sigma = \psi_* v'$ be the smallest simplex in $\mathcal{D}(Y)$ containing $v'$. Then $\sigma$ corresponds by Remark 3.3.10 to a cone $\sigma_{n,1}$ from Lemma 3.2.3 with $m = 1$ and $n = \dim\sigma$. Let $w_0, w_1, \ldots, w_n \in \sigma_{n,1}$ be the minimal generators of the cone $\sigma_{n,1}$, i.e. $w_i = e_1 + f_i$ in the notation of Lemma 3.2.3 (with $f_0 = 0$). The vertex $v'$ corresponds to a ray in the fan obtained by the resolution associated to $\psi$, see also the proof of Lemma 3.3.9. Let $w'$ denote the minimal generator of this ray. Since this fan is a toric subdivision of the cone $\sigma_{n,1}$ in the sense of [CLS11, before Example 3.3.12], we can write

$$w' = \sum_{i=0}^{n} a_i w_i$$

for some $a_i \geq 0$. As $\sigma_{n,1}$ (or equivalently $\sigma$) is simplicial, these $a_i$'s are unique. The $a_i$'s additionally satisfy

$$\sum_{i=0}^{n} a_i = 1, \tag{3.4.3}$$

by Proposition 3.2.6. We define for any vertex $v \in \mathcal{D}(Y)(0)$

$$d(v', v) = \begin{cases} r(1 - a_i) & \text{if } v = v_i, \\ r & \text{otherwise,} \end{cases} \tag{3.4.4}$$

where $v_i \in \mathcal{D}(Y)(0)$ is the vertex associated to the minimal generator $w_i$ in $\sigma_{n,1}$. It remains to check that $d(v', v) \in \mathbb{Z}_{\geq 0}$ for all $v \in \mathcal{D}(Y)(0)$. If $v \neq v_i$ for all $i = 0, 1, \ldots, n$, this is clear. For $i = 0, 1, \ldots, n$, we note that $0 \leq a_i \leq 1$, by (3.4.3). Hence $d(v', v_i) \geq 0$ for all $i = 0, 1, \ldots, n$. Moreover, the point $w'$ is a lattice point in the sublattice generated by $e_1, \frac{1}{r}f_1, \frac{1}{r}f_2, \ldots, \frac{1}{r}f_n$, see Proposition 3.2.6. Since $w_j = e_1 + f_j$, this immediately implies $a_i \in \frac{1}{r}\mathbb{Z}$. These two observations yield $d(v', v_i) \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, n$, i.e. $d$ is a well-defined map of the form (3.4.2).

**Construction 3.4.2.** We construct a function

$$I \colon \operatorname{RelWall}(\psi) \times \mathcal{D}(\tilde{Y})(0) \longrightarrow \mathbb{Z}, \tag{3.4.5}$$

which encodes the intersection numbers of one-cycles corresponding to the relative walls with the divisors corresponding to the vertices of the simplicial complex.

Let $\sigma$ be a simplex in $\mathcal{D}(Y)$ and let $\tau' = \sigma_1' \cap \sigma_2'$ be a wall of the subcomplex $\psi^*\sigma$ of $\mathcal{D}(\tilde{Y})$, i.e. $\tau'$ is a relative wall of $\psi$. Let $d - 1 = \dim \tau'$ be the dimension of the simplex $\tau'$. Let $v_0', \ldots, v_d'$ and $v_1', \ldots, v_{d+1}'$ be the vertices of $\sigma_1'$ and $\sigma_2'$, respectively. Recall once more that we associate to $\sigma$ the simplicial cone $\sigma_{d,1}$ from Lemma 3.2.3 with $m = 1$, see Remark 3.3.10. Then $\psi^*\sigma$ corresponds to a fan $\Sigma'$ which is a toric subdivision of the cone $\sigma_{\dim \sigma,1}$. The simplices $\sigma_1', \sigma_2'$, and $\tau'$ correspond to cones of $\Sigma'$ of dimensions $d+1, d+1$, and $d$ respectively. We denote the cones by $S_1'$, $S_2'$, and

$T'$ respectively. Note that $T' = S'_1 \cap S'_2$. Recall also that the vertices $v'_0, \ldots, v'_{d+1}$ correspond to rays $\rho'_0, \ldots, \rho'_{d+1}$ of $S'_1$ or $S'_2$. We define for any $v' \in \mathcal{D}(\tilde{Y})(0)$

$$I_{\tau'}(v') \coloneqq I(\tau', v') \coloneqq \begin{cases} D_{\rho'_i} \cdot V(T') & \text{if } v' = v'_i \\ 0 & \text{otherwise,} \end{cases} \tag{3.4.6}$$

where $D_{\rho'_i}$ is the divisor associated to the ray $\rho'_i$ and $V(T')$ is the Zariski closure of the orbit corresponding to $T'$ via the orbit-cone correspondence, see also Proposition 3.2.1. It is obvious that $I_{\tau'}(v')$ is an integer, i.e. the function $I$ is well-defined as claimed in (3.4.5).

These two auxiliary functions satisfy the following crucial relation, which is deduced from the wall relation, see (3.2.2).

**Lemma 3.4.3.** *The functions $d$ defined in Construction 3.4.1 and $I$ defined in Construction 3.4.2 satisfy for every $v \in \mathcal{D}(Y)(0)$ and $\tau' \in \mathrm{RelWall}(\psi)$*

$$\sum_{v' \in \mathcal{D}(\tilde{Y})(0)} (r - d(v', v))\, I_{\tau'}(v') = 0. \tag{3.4.7}$$

*Proof.* With the same notation as in Construction 3.4.2, let $w'_0, \ldots, w'_{d+1}$ be the minimal generators of the rays $\rho'_0, \ldots, \rho'_{d+1}$. The fan $\Sigma'$ is a toric subdivision of the cone $\sigma_{d,1}$ associated to $\sigma$ and let $w_1, \ldots, w_{d+1}$ be the minimal generators of the cone. We write for $i = 0, \ldots, d+1$ the minimal generators $w'_i$ as a linear combination

$$w'_i = \sum_{j=1}^{d+1} a_{ij} w_j, \tag{3.4.8}$$

which uniquely exists, since $\sigma'_1$ and $\sigma'_2$ are contained in the simplex $\sigma$. Moreover, by the wall relation (3.2.2) there exist (up to scaling) unique $b_0, \ldots, b_{d+1}$ such that

$$\sum_{i=0}^{d+1} b_i w'_i = 0.$$

Together with (3.4.8) we find

$$0 = \sum_{i=0}^{d+1} b_i w'_i = \sum_{i=0}^{d+1} b_i \left( \sum_{j=1}^{d+1} a_{ij} w_j \right) = \sum_{j=1}^{d+1} \left( \sum_{i=0}^{d+1} a_{ij} b_i \right) w_j$$

$$= \lambda \sum_{j=1}^{d+1} \left( \sum_{i=0}^{d+1} (r - d(v_j, v_i')) I_{\tau'}(v_i') \right) w_j,$$

where $\lambda = \frac{b_0}{r} \neq 0$ is the necessary scaling due to the choice with regard to the $b_j$'s. Note that we used in the last step the definition of $d$ and $I$ together with (3.2.3). Since $I_{\tau'}(v') = 0$ for every $v'$ different to all $(v_i')$'s, this shows (3.4.7) as the cone $\sigma$ is simplicial and $\lambda \neq 0$. □

In order to relate the homomorphism (3.4.1) with the homomorphism for the resolution $\tilde{X}$ after a finite base change, we replace the term $\bigoplus_j \mathrm{CH}_1(Y_j, \Lambda)$ in (3.4.1) with $\mathrm{CH}_1(Y, \Lambda)$ which does not change the image of the map, see Remark 3.4.5.

**Definition 3.4.4** ([PS23, Definition 3.1])**.** Let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable $R$-scheme with special fibre $Y$. Denote the irreducible components of $Y$ by $Y_i$ with $i \in I$. Then we define

$$\Phi^{\Lambda}_{\mathcal{X}, Y_i} \colon\ \mathrm{CH}_1(Y, \Lambda) \xrightarrow{\iota_*} \mathrm{CH}_1(\mathcal{X}, \Lambda) \xrightarrow{\iota_i^*} \mathrm{CH}_0(Y_i, \Lambda),$$

where $\iota \colon Y \to \mathcal{X}$ and $\iota_i \colon Y_i \to \mathcal{X}$ are the natural inclusions. Moreover, we define by abuse of notation (see Remark 3.4.5 below)

$$\Phi^{\Lambda}_{\mathcal{X}} \coloneqq \sum_{i \in I} \Phi^{\Lambda}_{\mathcal{X}, Y_i} \colon\ \mathrm{CH}_1(Y, \Lambda) \longrightarrow \bigoplus_{i \in I} \mathrm{CH}_0(Y_i, \Lambda).$$

**Remark 3.4.5.** The map in (3.4.1) is the composition of the natural surjection

$$\bigoplus_{i \in I} \mathrm{CH}_1(Y_i, \Lambda) \longrightarrow \mathrm{CH}_1(Y, \Lambda),$$

given by the pushforward along the inclusions $Y_i \hookrightarrow Y$, with the map $\Phi^{\Lambda}_{\mathfrak{X}}$ (defined in Definition 3.4.4), see also [PS23, Lemma 3.2]. In particular, the images of both maps named $\Phi^{\Lambda}_{\mathfrak{X}}$ are equal, which justifies the notation. Moreover, in the following we will mostly write $\Phi^{\Lambda}_{\mathfrak{X}}$ for the map in Definition 3.4.4.

**Remark 3.4.6.** Let the notation be as in Definition 3.4.4. Let $\gamma_i \in \mathrm{CH}_1(Y_i, \Lambda)$ be a one-cycle for some $i \in I$ and let $\iota_i' \colon Y_i \hookrightarrow Y$ be the natural inclusion. Then for every $j \in I \setminus \{i\}$

$$\Phi^{\Lambda}_{\mathcal{X}, Y_j}\left(\iota'_{i\,*}\gamma_i\right) = \begin{cases} \iota_{ij,j\,*}\ \gamma_i|_{Y_i \cap Y_j} \in \mathrm{CH}_0(Y_j, \Lambda), & \text{if } Y_i \cap Y_j \neq \emptyset, \\ 0 \in \mathrm{CH}_0(Y_j, \Lambda), & \text{otherwise,} \end{cases}$$

and

$$\Phi^{\Lambda}_{\mathcal{X},Y_i}(\iota'_{i*}\gamma_i) = -\sum_{\substack{l\neq i\\ Y_i\cap Y_l\neq\emptyset}} \iota_{il,i*}\,\gamma_i|_{Y_i\cap Y_l} \in \mathrm{CH}_0(Y_i,\Lambda),$$

where $\iota_{il,i}\colon Y_i\cap Y_l \hookrightarrow Y_i$ is the natural inclusion and we set $\gamma_i|_{Y_i\cap Y_j} := \iota^*_{ij,i}\gamma_i$, see [PS23, Lemma 3.2].

Recall that we fixed throughout this section a strictly semi-stable family $\mathfrak{X}\to \operatorname{Spec} R$ over a dvr $R$ and a resolution $\tilde{\mathfrak{X}}\to\operatorname{Spec}\tilde{R}$ after a finite extension of dvr's $\tilde{R}/R$ with induced extension $L/k$ of residue fields. We denote the special fibres by $Y$ and $\tilde{Y}$ and their dual complexes by $\mathcal{D}(Y)$ and $\mathcal{D}(\tilde{Y})$, respectively. Recall that $\psi\colon\mathcal{D}(\tilde{Y})\to\mathcal{D}(Y)$ is a subdivision, see Lemma 3.3.9. Recall that we denote the stratum associated to a simplex $\sigma$ in $\mathcal{D}(Y)$ by $Y_\sigma$.

**Lemma 3.4.7.** *The homomorphism $\Phi_{\mathcal{X}}$ from Definition 3.4.4 is given for the strictly semi-stable families $\mathfrak{X}\to\operatorname{Spec} R$ and $\tilde{\mathfrak{X}}\to\operatorname{Spec}\tilde{R}$ as follows:*

*(a) For any $\gamma\in\mathrm{CH}_1(Y,\Lambda)$ of the form (3.3.2) and for any $v\in\mathcal{D}(Y)(0)$,*

$$\Phi^{\Lambda}_{\mathfrak{X},Y_v}(\gamma) = \sum_{w\in A_{\mathcal{D}(Y)}(v)} \iota_{vw,v*}\left(\gamma_w|_{Y_v\cap Y_w} - \gamma_v|_{Y_v\cap Y_w}\right) \in \mathrm{CH}_0(Y_v,\Lambda),$$

*where $\iota_{vw,v}\colon Y_v\cap Y_w\hookrightarrow Y_v$ is the natural inclusion and $A_{\mathcal{D}(Y)}(v)$ is defined in Definition 3.3.2.*

*(b) For any $\tilde{\gamma}\in\mathrm{CH}_1(\tilde{Y},\Lambda)$ of the form (3.3.3) and for any $\tilde{v}\in\mathcal{D}(\tilde{Y})(0)$,*

$$\begin{aligned}\Phi^{\Lambda}_{\tilde{\mathfrak{X}},\tilde{Y}_{\tilde{v}}}(\tilde{\gamma}) = &\sum_{v\in A_\psi(\tilde{v})} -\tilde{\iota}_{v\tilde{v},\tilde{v}*}\,\tilde{\gamma}_{\psi_*\tilde{v}}|_{(Y_v\cap Y_{\psi_*\tilde{v}})_L} + \sum_{(\sigma,v)\in\mathcal{L}_\psi(\tilde{v})} \tilde{\iota}_{\sigma,\tilde{v}*}\,\tilde{\gamma}_v|_{Y_{\sigma,L}}\\ &+\sum_{\tau\in\mathrm{RelWall}(\psi)} \tilde{\iota}_{\tau,\tilde{v}*} I_\tau(\tilde{v})\tilde{\alpha}_\tau\end{aligned} \tag{3.4.9}$$

*in $\mathrm{CH}_0(\tilde{Y}_{\tilde{v}},\Lambda)$, where $A_{\mathcal{D}(\tilde{Y})}(v)$ is defined in Definition 3.3.2, $\psi_*$ in Definition 3.3.4, $A_\psi(\tilde{v})$, $\mathrm{RelWall}(\psi)$, and $\mathcal{L}_\psi(\tilde{v})$ in Definition 3.3.6, and $I_\tau(\tilde{v})$ in (3.4.6). Moreover, $\tilde{\iota}_{v\tilde{v};\tilde{v}}$, $\tilde{\iota}_{\sigma,\tilde{v}}$, and $\tilde{\iota}_{\tau,\tilde{v}}$ are the natural inclusions, see also the remark below.*

**Remark 3.4.8.** (a) The zero-cycles appearing on the right hand side of (3.4.9) lie in $(Y_v\cap Y_{\psi_*\tilde{v}})_L$ for $v\in A_\psi(\tilde{v})$, in $Y_{\sigma,L}$ for $(\sigma,v)\in\mathcal{L}_\psi(\tilde{v})$, or in $Y_{\psi_*\tau,L}$ for $\tau\in\mathrm{RelWall}(\psi)$. If $I_\tau(\tilde{v})\neq 0$, then we know by the construction in (3.4.6) that $\tilde{v}$ is a vertex of the simplex $\tau$. We disregard the terms $\alpha_\tau$, if $I_\tau(\tilde{v})=0$.

We know that a finite sequence of blow-ups in smooth centers and (Zariski) locally trivial $\mathbb{P}^1$-bundles of $(Y_v \cap Y_{\psi(\tilde{v})})_L$, $Y_{\sigma,L}$, or $Y_{\psi(\tau),L}$ (for $I_\tau(\tilde{v}) \neq 0$) lies in $\tilde{Y}_{\tilde{v}}$, see Proposition 3.3.12. These are the natural inclusions $\tilde{\iota}_{v\tilde{v},\tilde{v}}$, $\tilde{\iota}_{\sigma,\tilde{v}}$, and $\tilde{\iota}_{\tau,\tilde{v}}$, respectively. Note that this makes sense as the Chow group of zero-cycles is invariant under blow-ups in smooth centers and (Zariski) locally trivial $\mathbb{P}^1$-bundles by [Ful98, Theorem 3.3 (b) and Proposition 6.7 (e)].

(b) In accordance with the previous remark, we restrict the one-cycle $\tilde{\gamma}_{\psi_*\tilde{v}}$ in (3.4.9) not to $(Y_v \cap Y_{\psi_*\tilde{v}})_L$, but rather the strict transform of $(Y_v \cap Y_{\psi_*\tilde{v}})_L$ under the sequence of blow-ups described in the previous remark. Note that $(Y_v \cap Y_{\psi_*\tilde{v}})_L$ is already a Cartier divisor in $Y_{\psi_*\tilde{Y},L}$. The same holds for $\tilde{\gamma}_v|_{Y_{\sigma,L}}$.

(c) Two explicit examples of the map $\Phi_{\tilde{\mathfrak{X}}}$ are provided in Section 3.6.

*Proof of Lemma 3.4.7.* Item (a) follows immediately from the explicit description of $\Phi^\Lambda_{\tilde{\mathfrak{X}}}$ in Remark 3.4.6 together with (3.3.2). For item (b), we can use the same argument, i.e. apply Remark 3.4.6 to the explicit description of the one-cycles in (3.3.3). The only non-trivial part is the intersection

$$\tilde{Y}_{\tilde{v}} \cdot \tilde{\iota}_* \tilde{\iota}'_{\tau\,*} \tilde{q}^*_\tau \tilde{\alpha}_\tau = \tilde{\iota}_{\tau,\tilde{v}\,*} I_\tau(\tilde{v}) \alpha_\tau, \tag{3.4.10}$$

where $\tilde{\iota}\colon \tilde{Y} \hookrightarrow \tilde{\mathfrak{X}}$ is the natural inclusion and $\tilde{Y}_{\tilde{v}}\cdot$ is the intersection with the Cartier divisor $\tilde{Y}_{\tilde{v}} \subset \tilde{\mathfrak{X}}$. Since $\alpha_\tau$ is a linear combination of rational equivalent classes of points, it suffices to show (3.4.10) for every point in $Y_{\psi(\tau),L}$. For each such point we can consider the étale local toric description of $\tilde{\mathfrak{X}}$ and find that $I_\tau(\tilde{v})$ is by (3.4.6) the intersection multiplicity at that point. Since this is independent of the point chosen, we obtain (3.4.10) and thus item (b). □

**Proposition 3.4.9** (Key formula)**.** *Let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable $R$-scheme with special fibre $Y$, let $\tilde{R}/R$ be a finite extension of dvr's of ramification index $r$ with induced extension $L/k$ of residue fields, and let $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ be a resolution of the base-change $\mathfrak{X}_{\tilde{R}}$ from Proposition 2.3.3 and let $\tilde{Y}$ be its special fibre. Let $q\colon \tilde{Y} \to Y$ be the natural morphism and let $\psi\colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$ be the subdivision in Lemma 3.3.9. Then for any $\tilde{\gamma} \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ and $v \in \mathcal{D}(Y)(0)$ we have*

$$\Phi^\Lambda_{\mathfrak{X},Y_v}(q_*\tilde{\gamma}) = \sum_{\tilde{v} \in \mathcal{D}(\tilde{Y})(0)} \iota_{\psi_*\tilde{v},v\,*}(r - d(\tilde{v},v)) q_* \Phi^\Lambda_{\tilde{\mathfrak{X}},\tilde{Y}_{\tilde{v}}}(\tilde{\gamma}) \in \mathrm{CH}_0(Y_v, \Lambda), \tag{3.4.11}$$

*where $d(\tilde{v}, v)$ is defined in (3.4.4) and $\iota_{\psi_*\tilde{v},v}\colon Y_{\psi_*\tilde{v}} \hookrightarrow Y_v$ is the natural inclusion for $d(\tilde{v}, v) < r$.*

*Proof.* Let $v \in \mathcal{D}(Y)(0)$ be a vertex corresponding to an irreducible component $Y_v$ of the special fibre $Y$ and let $\tilde{\gamma} \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ be a one-cycle on $\tilde{Y}$ which we can assume to be of the form (3.3.3). Note first that $d(\tilde{v}, v) < r$ if and only if $v$ is a vertex of $\psi_* \tilde{v}$ by the construction in (3.4.4). Hence the right hand side of (3.4.11) is well-defined in $\mathrm{CH}_0(Y_v, \Lambda)$. We claim that the equality in (3.4.11) is an equality of cycles. If $r = 1$, then there is nothing to prove, as the base-change and the resolution are trivial, i.e. we can assume without loss of generality $r > 1$. We note that by the projection formula [Ful98, Proposition 2.3 (c)] for every $w \in A_{\mathcal{D}(Y)}(v)$

$$q_* \left( \tilde{\gamma}_v|_{(Y_v \cap Y_w)_L} \right) = (q_* \tilde{\gamma}_v)|_{Y_v \cap Y_w} . \tag{3.4.12}$$

Similarly, for every $\tilde{v} \in A_{\mathcal{D}(\tilde{Y})}(v)$ and $\sigma \in \mathcal{D}(Y)(1)$ such that $(\sigma, v) \in \mathcal{L}_\psi(\tilde{v})$ the projection formula yields

$$q_* \left( \tilde{\gamma}_v|_{Y_{\sigma,L}} \right) = (q_* \tilde{\gamma}_v)|_{Y_\sigma} . \tag{3.4.13}$$

Recall from (3.4.9) that $\Phi^\Lambda_{\tilde{\mathfrak{X}}, \tilde{Y}_{\tilde{v}}}(\tilde{\gamma})$ consists of three sums. We consider the right hand side of (3.4.11) for each of the sums separately: The terms $\tilde{\iota}_{\tau,v\,*} q_* \tilde{\alpha}_\tau$ in (3.4.9) sum up on the right side of (3.4.11) to 0 by Lemma 3.4.3. The right hand side of (3.4.11) reads for the first sum of (3.4.9) as follows (using the notation from Remark 3.4.8(a))

$$\begin{aligned}
&\sum_{\tilde{v} \in \mathcal{D}(\tilde{Y})(0)} (r - d(\tilde{v}, v)) \iota_{\psi_* \tilde{v}, v} q_* \left( \sum_{w \in A_\psi(\tilde{v})} -\tilde{\iota}_{w\tilde{v}, \tilde{v}\,*}\ \tilde{\gamma}_{\psi_* \tilde{v}}|_{(Y_w \cap Y_{\psi_* \tilde{v}})_L} \right) \\
&= \sum_{w \in A_{\mathcal{D}(Y)}(v)} -r q_* \tilde{\iota}_{w\psi^* v, \psi^* v\,*}\ \tilde{\gamma}_v|_{(Y_v \cap Y_w)_L} \\
&\overset{(3.4.12)}{=} \sum_{w \in A_{\mathcal{D}(Y)}(v)} -r \iota_{vw, v\,*}\ (q_* \tilde{\gamma}_v)|_{(Y_v \cap Y_w)} .
\end{aligned}$$

For second sum of (3.4.9) it reads (using again the notation from Remark 3.4.8(a))

$$\begin{aligned}
&\sum_{\tilde{v} \in \mathcal{D}(\tilde{Y})(0)} (r - d(\tilde{v}, v)) \iota_{\psi_* \tilde{v}, v} q_* \sum_{(\sigma, w) \in \mathcal{L}_\psi(\tilde{v})} \tilde{\iota}_{\sigma, \tilde{v}\,*}\ \tilde{\gamma}_w|_{Y_{\sigma,L}} \\
&= \sum_{w \in \mathcal{D}(Y)(0)} \sum_{\substack{\sigma \in \mathcal{D}(Y)(1) \\ w \in \sigma}} (r - d(\tilde{v}(w, \sigma), v)) \iota_{\psi_* \tilde{v}(w,\sigma), v\,*} q_* \tilde{\iota}_{\sigma, \tilde{v}(w,\sigma)\,*}\ \tilde{\gamma}_w|_{Y_{\sigma,L}} \\
&\overset{(3.4.13)}{=} \sum_{w \in \mathcal{D}(Y)(0)} \sum_{\substack{\sigma \in \mathcal{D}(Y)(1) \\ w \in \sigma}} (r - d(\tilde{v}(w, \sigma), v)) \iota_{\sigma, v\,*}\ (q_* \tilde{\gamma}_w)|_{Y_\sigma}
\end{aligned}$$

$$= \sum_{\substack{\sigma\in\mathcal{D}(Y)(1)\\ v\in\sigma}} (r-1)\iota_{\sigma,v\,*}\,(q_*\tilde{\gamma}_v)|_{Y_\sigma} + \iota_{\sigma,v\,*}\left(q_*\tilde{\gamma}_{w(\sigma)}\right)\Big|_{Y_\sigma}$$
$$= \sum_{w\in A_{\mathcal{D}(Y)}(v)} (r-1)\iota_{vw,v\,*}\,(q_*\tilde{\gamma}_v)|_{Y_v\cap Y_w} + \iota_{vw,v\,*}\,(q_*\tilde{\gamma}_w)|_{Y_v\cap Y_w}\,,$$

where $\tilde{v}(w,\sigma)\in\mathcal{D}(\tilde{Y})(0)$ is the unique vertex of $\mathcal{D}(Y)$ which is contained in $\sigma$ and adjacent to $\psi^*w$, $w(\sigma)$ is the unique vertex of the 1-simplex $\sigma$ different from $v$. Note that we used in the second last equality the easy observation that $d(\tilde{v}(w,\sigma),v)$ is 0 if $v\notin\sigma$, $r-1$ if $v$ is contained in $\sigma$ and adjacent to $\tilde{v}(w,\sigma)$, and 1 otherwise, which follows directly from Construction 3.4.1. Since $\Phi^\Lambda_{\mathfrak{X},Y_v}(q_*\tilde{\gamma})$ is given by

$$\Phi^\Lambda_{\mathfrak{X},Y_v}(q_*\tilde{\gamma}) = \sum_{w\in A_{\mathcal{D}(Y)}(v)} \iota_{vw,v\,*}\left((q_*\tilde{\gamma}_w)|_{Y_v\cap Y_w} - (q_*\tilde{\gamma}_v)|_{Y_v\cap Y_w}\right),$$

the above arguments shows the equality in (3.4.11). $\square$

## 3.5. Proof of the main results

### 3.5.1. Exactness of the complex

**Theorem 3.5.1.** *Let $R$ be a dvr with algebraically closed residue field $k$. Let $\mathfrak{X}\to\operatorname{Spec}R$ be a strictly semi-stable $R$-scheme with special fibre $Y$, let $\tilde{R}/R$ be a finite extension of dvr's with ramification index $r$, and let $\tilde{\mathfrak{X}}\to\operatorname{Spec}\tilde{R}$ be a resolution of $\mathfrak{X}_{\tilde{R}}$ from Proposition 2.3.3. Let $\Lambda$ be a ring and let $\psi\colon\mathcal{D}(\tilde{Y})\to\mathcal{D}(Y)$ be the subdivision from Lemma 3.3.9. Let $A/R$ and $\tilde{A}/\tilde{R}$ be unramified extensions of dvr's with the same induced extension $L'/k$ on residue fields. Then the following holds*

*(a) If $\Phi^\Lambda_{\tilde{\mathfrak{X}}_{\tilde{A}}}$ is surjective, then so is $\Phi^\Lambda_{\mathfrak{X}_A}$.*

*(b) Assume additionally that $\mathfrak{X}$ is proper. If the complex*

$$\mathrm{CH}_1(\tilde{Y}_{L'},\Lambda)\xrightarrow{\Phi^\Lambda_{\tilde{\mathfrak{X}}_{\tilde{A}}}}\bigoplus_{\tilde{v}\in\mathcal{D}(\tilde{Y})(0)}\mathrm{CH}_0(\tilde{Y}_{\tilde{v},L'},\Lambda)\xrightarrow{\sum_{\tilde{v}}\deg}\Lambda \tag{3.5.1}$$

*is exact, then the complex*

$$\mathrm{CH}_1(Y_{L'},\Lambda)\xrightarrow{\Phi^\Lambda_{\mathfrak{X}_A}}\bigoplus_{v\in\mathcal{D}(Y)(0)}\mathrm{CH}_0(Y_{v,L'},\Lambda)\xrightarrow{\sum_v\deg}\Lambda \tag{3.5.2}$$

*is exact.*

**Remark 3.5.2.** Let $R$ be a discrete valuation ring with residue field $k$ and let $L/k$ be a field extension. Then there exists an unramified extension of discrete valuation rings $A/R$ such that the induced extension on residue field is $L/k$, see [Bou06, Théorème 1 and Corollaire on page AC IX.40-41].

*Proof of Theorem 3.5.1.* Note that the families $\mathfrak{X}_A \to \operatorname{Spec} A$ and $\tilde{\mathfrak{X}}_{\tilde{A}} \to \operatorname{Spec} \tilde{A}$ are strictly semi-stable by [Har01, Proposition 1.3], as $A/R$ and $\tilde{A}/\tilde{R}$ are unramified extension. Moreover, the natural morphism $q\colon \tilde{Y} \to Y$ induces a morphism of the base-change $q\colon \tilde{Y}_{L'} \to Y_{L'}$. We will prove the statement for ease of notation in the case $L' = k$. The general case follows verbatim by the same argument.

We claim: If $\Phi^{\Lambda}_{\tilde{\mathfrak{X}}}$ is surjective or $\mathfrak{X}$ is proper and the complex (3.5.1) is exact, then

$$q_* \Phi^{\Lambda}_{\tilde{\mathcal{X}}} \left( \mathrm{CH}_1(\tilde{Y}, \Lambda) \right) = \Phi^{\Lambda}_{\mathcal{X}} \left( \mathrm{CH}_1(Y, \Lambda) \right), \tag{3.5.3}$$

where $q\colon \tilde{Y} \to Y$ is the natural morphism induced by the morphism $\tilde{\mathfrak{X}} \to \mathfrak{X}$.

The claim immediately implies items (a) + (b), as the push-forward map

$$q_*\colon \bigoplus_{\tilde{v} \in \mathcal{D}(\tilde{Y})(0)} \mathrm{CH}_0(\tilde{Y}_{\tilde{v}}, \Lambda) \longrightarrow \bigoplus_{v \in \mathcal{D}(Y)(0)} \mathrm{CH}_0(Y_v, \Lambda) \tag{3.5.4}$$

is clearly surjective, see Proposition 3.3.12, and $q_*$ commutes with the degree map. Hence it suffices to prove the claim. We show first the inclusion "$\supseteq$" in (3.5.3). Let $\gamma \in \mathrm{CH}_1(Y, \Lambda)$ be a one-cycle. We aim to show that $\Phi^{\Lambda}_{\mathfrak{X}}(\gamma)$ lies in the image of $q_* \Phi^{\Lambda}_{\tilde{\mathfrak{X}}}$. It follows from Proposition 3.3.14 that there exists a $\tilde{\gamma} \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ such that

$$q_* \tilde{\gamma} = \gamma \in \mathrm{CH}_1(Y).$$

(Note that we used here that $k$ is the residue field of $R$ and $\tilde{R}$, as $k$ is algebraically closed.) The key formula (3.4.11) in Proposition 3.4.9 shows

$$\Phi^{\Lambda}_{\mathfrak{X}}(\gamma) = \Phi^{\Lambda}_{\mathfrak{X}}(q_* \tilde{\gamma}) \in q_* \Phi^{\Lambda}_{\tilde{\mathfrak{X}}}.$$

This concludes the proof of the inclusion "$\supseteq$". We turn to the proof of the inclusion "$\subseteq$" in (3.5.3). Consider a collection of zero-cycles $\beta = (\beta_v)_v \in \bigoplus_{v \in \mathcal{D}(Y)(0)} \mathrm{CH}_0(Y_v, \Lambda)$ of the form

$$\beta_v = \begin{cases} \iota_{\sigma, v_1 *} \alpha & \text{if } v = v_1, \\ -\iota_{\sigma, v_2 *} \alpha & \text{if } v = v_2, \\ 0 & \text{otherwise}, \end{cases} \tag{3.5.5}$$

for some $\sigma \in \mathcal{D}(Y)(1)$ with vertices $v_1$ and $v_2$, $\alpha \in \mathrm{CH}_0(Y_\sigma, \Lambda)$ and $\iota_{\sigma,v}\colon Y_\sigma \hookrightarrow Y_v$ the natural inclusion. We aim to show that $\beta \in \Phi^\Lambda_{\mathfrak{X}}(\mathrm{CH}_1(Y,\Lambda))$. To this end, consider $\tilde{\gamma} \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ such that

$$\left(\Phi^\Lambda_{\tilde{\mathfrak{X}},\tilde{Y}_{\tilde{v}}}\right)(\tilde{\gamma}) = \begin{cases} \tilde{\iota}_{\sigma,\psi^* v_1 *}\alpha & \text{if } \tilde{v} = \psi^* v_1, \\ -\alpha & \text{if } (\sigma, v_1) \in \mathcal{L}_\psi(\tilde{v}), \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{\iota}_{\sigma,\psi^* v_1}$ is as in Remark 3.4.8(a) and $\mathcal{L}_\psi(\tilde{v})$ is defined in Definition 3.3.6(iii). Note that there exists a unique vertex $\tilde{v}$ such that $(\sigma, v_1) \in \mathcal{L}_\psi(\tilde{v})$. Hence such a $\tilde{\gamma}$ exists by our assumption that $\Phi^\Lambda_{\tilde{\mathfrak{X}}}$ is surjective or $\mathfrak{X}$ is proper and the complex (3.5.1) is exact. Then the formula (3.4.11) implies

$$\begin{aligned}
\left(\Phi^\Lambda_{\mathfrak{X},Y_{v_1}}\right)(q_*\tilde{\gamma}) &= r\iota_{\sigma,v_1 *}\alpha + \iota_{\sigma,v_1 *}(r-1)(-\alpha) + 0 = \iota_{\sigma,v_1 *}\alpha, \\
\left(\Phi^\Lambda_{\mathfrak{X},Y_{v_2}}\right)(q_*\tilde{\gamma}) &= 0 + 1 \cdot \iota_{\sigma,v_2 *}(-\alpha) = -\iota_{\sigma,v_2 *}\alpha, \\
\left(\Phi^\Lambda_{\mathfrak{X},Y_{v}}\right)(q_*\tilde{\gamma}) &= 0,
\end{aligned}$$

for $v \in \mathcal{D}(Y) \setminus \{v_1, v_2\}$. Hence, $\beta = \Phi^\Lambda_{\mathfrak{X}}(q_*\tilde{\gamma}) \in \Phi^\Lambda_{\mathcal{X}}(\mathrm{CH}_1(Y,\Lambda))$.

Since $q_*\Phi^\Lambda_{\tilde{\mathfrak{X}}}(\tilde{\gamma}')$ for every $\tilde{\gamma}' \in \mathrm{CH}_1(\tilde{Y}, \Lambda)$ is a (finite) linear combination of zero-cycles of the form (3.5.5) by Remark 3.4.6, the above argument shows that

$$q_*\Phi^\Lambda_{\tilde{\mathcal{X}}}\left(\mathrm{CH}_1(\tilde{Y},\Lambda)\right) \subseteq \Phi^\Lambda_{\mathcal{X}}(\mathrm{CH}_1(Y,\Lambda)).$$

Hence, we proved the equality (3.5.3), which finishes the proof of the theorem. □

**Remark 3.5.3.** Note that the assumption on the residue field being algebraically closed is crucial for the argument. Because otherwise, the push-forward map (3.5.4) is in general not surjective. In fact, for a vertex $v \in \mathcal{D}(Y)(0)$ and $z \in \mathrm{CH}_0(Y_v, \Lambda)$, we have

$$q_* z_L = [L : k] \cdot z \in \mathrm{CH}_0(Y_v, \Lambda),$$

where $L$ is the residue field of $\tilde{R}$.

Using the same argument as in [PS23, Theorem 4.1], Theorem 3.5.1(b) implies the following result.

**Corollary 3.5.4.** *Let $\Lambda$ be a ring. Let $R$ be a dvr with algebraically closed residue field $k$, let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable projective $R$-scheme with special fibre*

*$Y = \bigcup_{i \in I} Y_i$ such that the geometric generic fibre $\bar{X}$ of $\mathfrak{X}$ admits a $\Lambda$-decomposition of the diagonal. Then for any unramified extension $A/R$ of dvr's with induced extension of residue fields $L'/k$, the complex*

$$\bigoplus_{i \in I} \mathrm{CH}_1\left(Y_{i,L'}, \Lambda\right) \xrightarrow{\Phi^{\Lambda}_{\mathfrak{X}_A}} \bigoplus_{i \in I} \mathrm{CH}_0\left(Y_{i,L'}, \Lambda\right) \xrightarrow{\deg} \Lambda \tag{3.5.6}$$

*is exact, where is $\Phi^{\Lambda}_{\mathfrak{X}_A}$ as in (3.4.1), see also Definition 3.4.4 and Remark 3.4.6.*

*Proof.* As $\Phi^{\Lambda}_{\mathfrak{X}}$ depends only on the special fibre (see [PS23, Lemma 3.2]) and the special fibre does not change under the base-change to the completion of $R$, we can assume without loss of generality that $R$ is complete. The $\Lambda$-decomposition of the diagonal for the geometric generic fibre holds already after some finite field extension $F/K$ of the fraction field of $R$.

Consider the integral closure $\tilde{R}$ of $R$ in $F$. Since $F$ is a finite field extension and $R$ is complete, the ring $\tilde{R}$ is also a dvr and a finite extension over $R$. Note that the residue field of $\tilde{R}$ is again $k$ because $k$ is algebraically closed. Moreover, fix a resolution $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ of the base-change $\mathfrak{X}_{\tilde{R}}$ from Proposition 2.3.3.

Let $A/R$ be a unramified extension of dvr's with induced extension $L'/k$ of residue fields. By Remark 3.5.2 there exists an unramified extension $\tilde{A}/\tilde{R}$ whose extension of residue fields is $L'/k$. Since the generic fibre $\tilde{X} = X_F$ of $\tilde{\mathfrak{X}}$ admits a $\Lambda$-decomposition of the diagonal, the homomorphism $\Phi^{\Lambda}_{\tilde{\mathfrak{X}}_{\tilde{A}}}$ is surjective onto the kernel of the degree map by [PS23, Theorem 1.2 (1)]. (*Loc. cit.* shows it for $\Lambda = \mathbb{Z}$, but the same argument works for general $\Lambda$.) Hence, the complex (3.5.6) is exact by Theorem 3.5.1(b) and Remark 3.4.5. □

*Proof of Theorem 3.1.1.* Let $R$ be a dvr with fraction field $k$ and let $L/k$ be a field extension. By [Bou06, Corollaire on page AC IX.41], there exists a unramified extension $A/R$ whose induced extension of residue fields is $L/k$. Thus Theorem 3.1.1 is a reformulation of Corollary 3.5.4, because the degree map $\sum_i \deg$ is clearly surjective as $k = \overline{k}$. □

The assumption on the residue field in Theorem 3.1.1 and Corollary 3.5.4 is crucial.

**Remark 3.5.5.** Theorem 3.1.1 and Corollary 3.5.4 show the exactness of the complex (3.1.1) for strictly semi-stable $R$-scheme if the residue field $k$ of $R$ is algebraically closed and the geometric generic fibre admits a decomposition of the diagonal.

The exactness of the complex (3.1.1) at $\mathbb{Z}$ requires the existence of a zero-cycle of degree 1 on the special fibre. Such a zero-cycle does not exist in general over non-closed fields. The exactness of the complex (3.1.1) fails over non-closed fields in general also at $\bigoplus_i \mathrm{CH}_0(Y_i)$. Indeed, consider a smooth, projective geometrically rational variety $Y$ over a field $k \neq \overline{k}$ such that $\mathrm{A}_0(Y) := \ker(\deg\colon \mathrm{CH}_0(Y) \to \mathbb{Z})$ is non-trivial. Then the smooth, projective family $\mathfrak{X} = Y \times_k k[[t]] \to \operatorname{Spec} k[[t]]$ is strictly semi-stable and the image of $\Phi$ is trivial. Hence (3.1.1) is not exact, but the geometric generic fibre is rational and thus admits a decomposition of diagonal. As an example for $Y$, we can consider the diagonal cubic surface $S_1 := \{x^3 + y^3 + z^3 + pw^3 = 0\} \subset \mathbb{P}^3_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ which satisfies $\mathrm{A}_0(S_1) \neq 0$ by [CTS96, Example 2.8] and [SS14, Theorem 4.1.1]. By [CT18, Proposition 1.3] there is a cubic surface $S_2$ over $\mathbb{R}$ with $\mathrm{A}_0(S_2) \neq 0$, which yields another example for $Y$.

We also provide a counterexample over number fields. Consider a (diagonal) cubic surface $S$ over a number field whose Brauer group $\mathrm{Br}(S)$ is non-trivial, see e.g. [CTKS87, Proposition 1] and [SD93]. This implies by [Mer08, Theorem 2.11] that there exists a field extension $L/k$ such that $\mathrm{A}_0(S_L) \neq 0$. Hence, the complex (3.1.1) is not exact after base-change to $L$.

### 3.5.2. Geometrically retract rational varieties over Laurent fields

We prove Corollary 3.1.2 more generally over fraction fields of excellent, henselian dvr's with algebraically closed residue field and for Chow groups with $\Lambda$-coefficients. Recall that the exponential characteristic of a field $k$ is $p$ if the characteristic $\operatorname{char} k = p > 0$ and 1 if $\operatorname{char} k = 0$.

**Corollary 3.5.6.** *Let $R$ be an excellent, henselian dvr with fraction field $K$ and algebraically closed residue field $k$. Let $X$ be a smooth, projective variety over $K$ and assume that $X$ admits a strictly semi-stable projective model over $R$. If $X$ admits geometrically a $\Lambda$-decomposition of the diagonal, then the degree map* $\deg\colon \mathrm{CH}_0(X,\Lambda) \to \Lambda$ *is an isomorphism up to inverting the exponential characteristic of $k$.*

*Proof.* We first note that the degree map is surjective. Indeed, let $\mathfrak{X} \to \operatorname{Spec} R$ be a strictly semi-stable projective $R$-scheme with generic fibre $X$. Since $k$ is algebraically closed, there exists a $k$-rational point in the smooth locus of the special fibre. By Hensel's lemma we can lift this $k$-point to a section of $\mathfrak{X} \to \operatorname{Spec} R$, see e.g. [EGAIV, Theorem 18.5.17]. The intersection of this section with $X$ yields a zero-cycle of degree 1 on $X$, i.e. the degree map is surjective.

By assumption, the base-change $\bar{X}$ to the algebraic closure $\bar{K}$ admits a $\Lambda$-decomposition of the diagonal. This $\Lambda$-decomposition of the diagonal holds after some finite field extension $F/K$. By a pull/push argument, we find that there exists an $N \in \mathbb{Z}_{\geq 1}$ such that

$$N[\Delta_X] = [X \times z] + [Z] \in \mathrm{CH}_{\dim X}(X \times_K X, \Lambda), \tag{3.5.7}$$

where $\Delta_X \subset X \times_K X$ is the diagonal, $z$ is a zero-cycle on $X$ of degree 1, and $Z$ is a $\dim X$-cycle on $X \times_K X$ which does not dominate the first factor. Hence the kernel of the degree map

$$\mathrm{A}_0(X, \Lambda) := \ker\left(\deg\colon \mathrm{CH}_0(X, \Lambda) \longrightarrow \Lambda\right)$$

is $N$-torsion, in fact one can choose $N = [F : K]$, see e.g. [ACTP17, proof of Lemma 1.3]. Let $Y$ denote the special fibre of the strictly semi-stable family $\mathfrak{X} \to \operatorname{Spec} R$ and let $Y_i$ denote the irreducible components of $Y$ with $i \in I$. Consider Fulton's localization exact sequence

$$\mathrm{CH}_1(Y, \Lambda) \longrightarrow \mathrm{CH}_1(\mathfrak{X}, \Lambda) \longrightarrow \mathrm{CH}_0(X, \Lambda) \longrightarrow 0,$$

see [Ful75, Section 4.4]. This sequence fits, after tensoring with $\mathbb{Z}/l$ for some $l \in \mathbb{Z}_{\geq 0}$ coprime to the exponential characteristic of $k$, by [SS10, Corollary 0.9] into the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_1(Y, \Lambda)/l \arrow[r] \arrow[d, equal] & \mathrm{CH}_1(\mathfrak{X}, \Lambda)/l \arrow[r] \arrow[d, "\simeq"] & \mathrm{CH}_0(X, \Lambda)/l \arrow[r] \arrow[d, "\deg"] & 0 \\
\mathrm{CH}_1(Y, \Lambda)/l \arrow[r, "\deg_i \circ \Phi^{\Lambda}_{\mathfrak{X}}"] & \left(\bigoplus_{i \in I} \Lambda/l \cdot [Y_i]\right)^{\vee} \arrow[r, "\deg"] & \Lambda/l \arrow[r] & 0,
\end{tikzcd} \tag{3.5.8}$$

see also Bloch's argument in [EW16, Theorem A.1] or [EKW16]. The bottom row is exact by Corollary 3.5.4 using that the map

$$\mathrm{CH}_0(Y_i, \Lambda)/l \xrightarrow{\deg_i} \left(\bigoplus_{i \in I} \Lambda/l \cdot [Y_i]\right)^{\vee}$$

is surjective, as $k$ is algebraically closed. Thus, the right arrow in (3.5.8) is injective, in particular $\mathrm{A}_0(X, \Lambda)$ is divisible by $l$ for each $l$ coprime to the exponential characteristic of $k$. Since $\mathrm{A}_0(X, \Lambda)$ is torsion by (3.5.7), the corollary follows. □

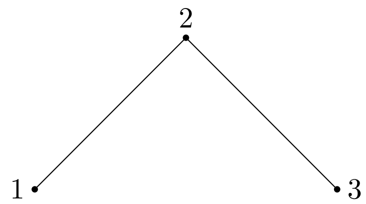
## 3.6. Two concrete resolutions

We illustrate the proof of the key proposition (Proposition 3.4.9) for $\Lambda = \mathbb{Z}$ in two examples by spelling out the constructions and arguments in Sections 3.3 and 3.4.

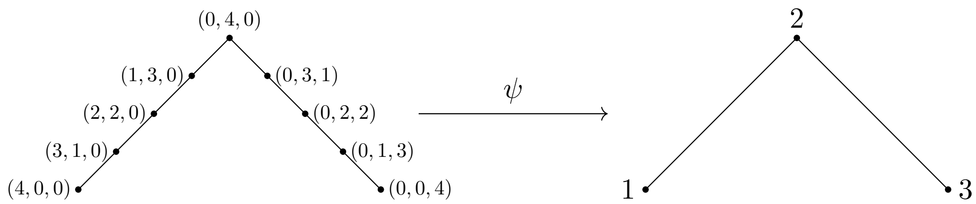
Recall the general setup from Setup 3.1.4: We consider a strictly semi-stable family $\mathfrak{X} \to \operatorname{Spec} R$ over a dvr with special fibre $Y$ such that the irreducible components of $Y$ and for simplicity all their intersections are geometrically integral. Let $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ be a resolution from Proposition 2.3.3 after a finite extension $\tilde{R}/R$ of dvr's of ramification index $r$ with induced extension $L/k$ of residue fields. We denote the special fibre of $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ by $\tilde{Y}$ and the natural morphism by $q \colon \tilde{Y} \to Y$.

To simplify the notation, we will omit the pushforward along the natural inclusions in all the formulas contained in the two examples below.

**Example 3.6.1.** Assume that $Y$ is a chain of three Cartier divisors and denote the three irreducible components by $Y_1, Y_2, Y_3$. The dual complex $\mathcal{D}(Y)$ of $Y$ from Definition 3.3.7 is




where we abbreviate the vertex corresponding to $Y_i$ by its subscript $i$. The subdivision from Lemma 3.3.9 is then given in the case $r = 4$ by




Recall that the subdivision $\psi \colon \mathcal{D}(\tilde{Y}) \to \mathcal{D}(Y)$ corresponds to a resolution $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ of $\mathfrak{X}_{\tilde{R}} \to \operatorname{Spec} \tilde{R}$. The irreducible components of the special fibre of $\tilde{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}$ correspond to the vertices $(i_1, i_2, i_3) \in \mathbb{N}^3$ of $\mathcal{D}(\tilde{Y})$: The components $\tilde{Y}_{(4,0,0)}$, $\tilde{Y}_{(0,4,0)}$, and $\tilde{Y}_{(0,0,4)}$ are isomorphic to $Y_{1,L}$, $Y_{2,L}$ and $Y_{3,L}$, respectively. The component $\tilde{Y}_{(i,_1,i_2,i_3)}$ corresponding to a vertex $(i_1, i_2, i_3)$ with $i_2, i_1 + i_3 \neq 0$ in the above picture is a (Zariski) locally trivial $\mathbb{P}^1$-bundle over $Y_{j,L} \cap Y_{2,L}$ where $j \in \{1, 3\}$ such that $i_j \neq 0$.

By Proposition 3.3.14, we have a surjection

$$f\colon \bigoplus_{j=1}^{3} \mathrm{CH}_1(Y_{j,L}) \oplus \bigoplus_{\substack{i_1,i_2\geq 1\\ i_1+i_2=4}} \mathrm{CH}_0((Y_1\cap Y_2)_L) \oplus \bigoplus_{\substack{j_2,j_3\geq 1\\ j_2+j_3=4}} \mathrm{CH}_0((Y_2\cap Y_3)_L) \longrightarrow \mathrm{CH}_1(\tilde{Y}),$$

$$(\gamma_1,\gamma_2,\gamma_3,\alpha_{(i_1,i_2,0)},\alpha_{(0,j_2,j_3)}) \mapsto \gamma_1+\gamma_2+\gamma_3+\sum_{\substack{i_1,i_2\geq 1\\ i_1+i_2=4}} q^*_{(i_1,i_2,0)}\alpha_{(i_1,i_2,0)} + \sum_{\substack{j_2,j_3\geq 1\\ j_2+j_3=4}} q^*_{(0,j_2,j_3)}\alpha_{(0,j_2,j_3)}$$

where $q_{(i_1,i_2,0)}\colon \tilde{Y}_{(i_1,i_2,0)} \to (Y_1\cap Y_2)_L$ and $q_{(0,j_2,j_3)}\colon \tilde{Y}_{(0,j_2,j_3)} \to (Y_2\cap Y_3)_L$ are the natural (flat) projections of the projective bundles. The distance function $d$ from Construction 3.4.1 can be read of the indices of the components as follows:

$$d((i_1,i_2,i_3),(4,0,0)) = 4-i_1, \quad d((i_1,i_2,i_3),(0,4,0)) = 4-i_2, \quad d((i_1,i_2,i_3),(0,0,4)) = 4-i_3.$$

Note that the relative walls in this examples are the new vertices $(i_1,i_2,0)$ and $(0,j_2,j_3)$ with $i_1,\ i_2,\ j_2,\ j_3>0$; hence we denote the relative walls by the vertex. Then the function $I_\tau(\tilde{v})$ from Construction 3.4.2 is given for $\tau=(2,2,0)$ by

$$I_{(2,2,0)}((i_1,i_2,i_3)) = \begin{cases} 1 & \text{if } (i_1,i_2,i_3)=(1,3,0) \text{ or } (3,1,0),\\ -2 & \text{if } (i_1,i_2,i_3)=(2,2,0),\\ 0 & \text{otherwise}\end{cases}$$

and similarly for the other relative walls. The explicit description of the morphism $\Phi_{\tilde{\mathfrak{X}}}$ from Lemma 3.4.7 is given as follows

$$\begin{aligned}
\Phi_{(4,0,0)}(\gamma) &= -\gamma_1|_{(Y_1\cap Y_2)_L} + \alpha_{(3,1,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(4,0,0)}),\\
\Phi_{(0,4,0)}(\gamma) &= -\gamma_2|_{(Y_1\cap Y_2)_L} - \gamma_2|_{(Y_2\cap Y_3)_L} + \alpha_{(0,3,1)} + \alpha_{(1,3,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,4,0)}),\\
\Phi_{(0,0,4)}(\gamma) &= -\gamma_3|_{(Y_2\cap Y_3)_L} + \alpha_{(0,1,3)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,0,4)}),\\
\Phi_{(3,1,0)}(\gamma) &= \gamma_1|_{(Y_1\cap Y_2)_L} + \alpha_{(2,2,0)} - 2\alpha_{(3,1,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(3,1,0)}),\\
\Phi_{(2,2,0)}(\gamma) &= \alpha_{(1,3,0)} + \alpha_{(3,1,0)} - 2\alpha_{(2,2,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(2,2,0)}),\\
\Phi_{(1,3,0)}(\gamma) &= \gamma_2|_{(Y_1\cap Y_2)_L} + \alpha_{(2,2,0)} - 2\alpha_{(1,3,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(1,3,0)}),\\
\Phi_{(0,3,1)}(\gamma) &= \gamma_2|_{(Y_2\cap Y_3)_L} + \alpha_{(0,2,2)} - 2\alpha_{(0,3,1)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,3,1)}),\\
\Phi_{(0,2,2)}(\gamma) &= \alpha_{(0,3,1)} + \alpha_{(0,1,3)} - 2\alpha_{(0,2,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,2,2)}),\\
\Phi_{(0,1,3)}(\gamma) &= \gamma_3|_{(Y_2\cap Y_3)_L} + \alpha_{(0,2,2)} - 2\alpha_{(0,1,3)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,1,3)}),
\end{aligned}$$

where $\Phi_{(i_1,i_2,i_3)} = \Phi_{\tilde{\mathfrak{X}},\tilde{Y}_{(i_1,i_2,i_3)}}$ and $\gamma = f\left(\gamma_1, \gamma_2, \gamma_3, \alpha_{(i_1,i_2,0)}, \alpha_{(0,j_2,j_3)}\right) \in \mathrm{CH}_1(\tilde{Y})$. The key formula (Proposition 3.4.9) reads in this example as follows:

$$
\begin{aligned}
\Phi_{\mathfrak{X},Y_1}(q_*\gamma) &= \sum_{i_1=1}^{4} i_1 \cdot q_*\Phi_{(i_1,4-i_1,0)}(\gamma),\\
\Phi_{\mathfrak{X},Y_2}(q_*\gamma) &= 4q_*\Phi_{(0,4,0)}(\gamma) + \sum_{i_2=1}^{3} i_2 \cdot q_*\Phi_{(4-i_2,i_2,0)}(\gamma) + i_2 \cdot q_*\Phi_{(0,i_2,4-i_2)}(\gamma),\\
\Phi_{\mathfrak{X},Y_3}(q_*\gamma) &= \sum_{j_3=1}^{4} j_3 \cdot q_*\Phi_{(0,4-j_3,j_3)}(\gamma).
\end{aligned}
$$

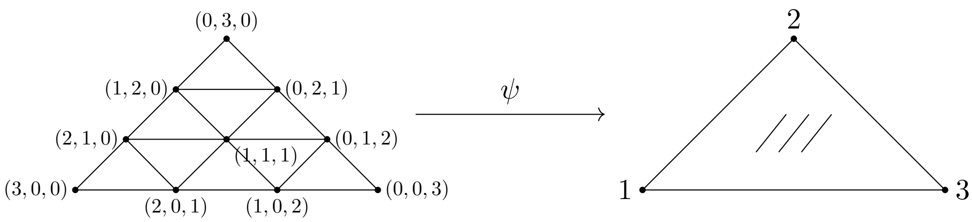
This can be checked directly from the above description of $\Phi_{\tilde{\mathfrak{X}}}$ using the observations

$$
(q_*\gamma_i)|_{Y_i\cap Y_j} = q_*\left(\gamma_i|_{(Y_i\cap Y_j)_L}\right), \quad q_*q^*_{(i_1,i_2,i_3)}\alpha_{(i_1,i_2,i_3)} = 0,
$$

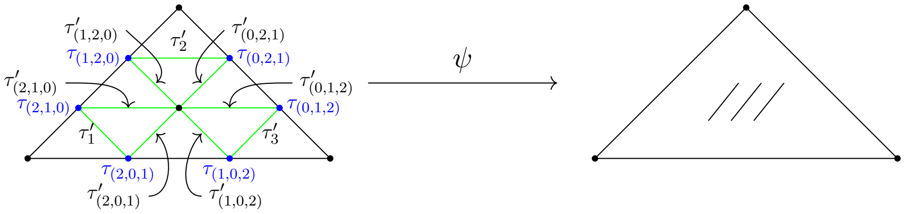
see [Ful98, Proposition 2.3 (c)] for the first claim. The explicit description of $\Phi_{\mathfrak{X}}$ in Lemma 3.4.7 reads in this example

$$
\begin{aligned}
\Phi_{\mathfrak{X},Y_1}(q_*\gamma) &= -\,(q_*\gamma_1)|_{Y_1\cap Y_2} + (q_*\gamma_2)|_{Y_1\cap Y_2}\,,\\
\Phi_{\mathfrak{X},Y_2}(q_*\gamma) &= -\,(q_*\gamma_2)|_{Y_1\cap Y_2} - (q_*\gamma_2)|_{Y_2\cap Y_3} + (q_*\gamma_1)|_{Y_1\cap Y_2} + (q_*\gamma_3)|_{Y_2\cap Y_3}\,,\\
\Phi_{\mathfrak{X},Y_3}(q_*\gamma) &= -\,(q_*\gamma_3)|_{Y_2\cap Y_3} + (q_*\gamma_2)|_{Y_2\cap Y_3}\,.
\end{aligned}
$$

**Example 3.6.2.** Assume that $Y$ is a complete intersection of three Cartier divisors $Y_1$, $Y_2$, and $Y_3$, i.e. the dual complex $\mathcal{D}(Y)$ of $Y$ is the standard 2-simplex. For $r = 3$, a possible subdivision looks like:




This subdivision can be obtained by the blow-ups along the components corresponding to the vertices $(3,0,0), (3,0,0), (0,3,0), (1,2,0), (0,3,0), (1,2,0), (2,1,0)$ in that order. The component $\tilde{Y}_{(1,1,1)}$ corresponding to the central vertex $(1,1,1)$ in the picture is a Zariski locally trivial $\mathbb{P}^1 \times \mathbb{P}^1$-bundle blown up along two disjoint sections. The components indexed by $(3,0,0)$, $(0,3,0)$, and $(0,0,3)$ are isomorphic to $Y_{1,L}$, $Y_{2,L}$, and $Y_{3,L}$, respectively. The other components are blow-ups of a Zariski locally trivial $\mathbb{P}^1$-bundle over the intersection $(Y_i \cap Y_j)_L$ of two components of $Y$ along the smooth locus $(Y_1 \cap Y_2 \cap Y_3)_L$ viewed via a section as a subvariety in the projective bundle. There are two types of relative walls which are depicted below in two different colors:




To simplify notation, we consider the following action of the symmetric group $S_3$ on the vertices: $\sigma(i_1, i_2, i_3) = (i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)})$ for every $(i_1, i_2, i_3) \in \mathbb{N}_0^3$. By Proposition 3.3.14, we can write any one-cycle $\gamma \in \mathrm{CH}_1(\tilde{Y})$ as

$$\gamma = \gamma_1+\gamma_2+\gamma_3+(q'_1)^*\alpha'_1+(q'_2)^*\alpha'_2+(q'_3)^*\alpha'_3+\sum_{\sigma\in S_3}\left(q_{\sigma(2,1,0)}\right)^*\alpha_{\sigma(2,1,0)}+\left(q'_{\sigma(2,1,0)}\right)^*\alpha'_{\sigma(2,1,0)}$$

in $\mathrm{CH}_1(\tilde{Y})$ for some $\gamma_i \in \mathrm{CH}_1(Y_{i,L})$, $\alpha_{(i_1,i_2,i_3)} \in \mathrm{CH}_0(\tilde{Y}_{(i_1,i_2,i_3)})$, and $\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_{(i_1,i_2,i_3)}$ in $\mathrm{CH}_0((Y_1 \cap Y_2 \cap Y_3)_L)$, where the zero-cycles $\alpha_{\dots}$ and $\alpha'_{\dots}$ correspond to the respective relative wall in the picture above and the morphisms $q_{\dots}$ and $q'_{\dots}$ are the natural projection of the associated $\mathbb{P}^1$-bundle. The distance function $d$ is given by

$$d((i_1,i_2,i_3),(4,0,0)) = 4-i_1, \quad d((i_1,i_2,i_3),(0,4,0)) = 4-i_2, \quad d((i_1,i_2,i_3),(0,0,4)) = 4-i_3.$$

We write down the function $I$ representing the intersection numbers by the matrix

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & -2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}$$

where the columns are labelled in the following order:

$$\tau_{(2,1,0)}, \tau_{(1,2,0)}, \tau_{(2,0,1)}, \tau_{(1,0,2)}, \tau_{(0,2,1)}, \tau_{(0,1,2)}, \tau'_1, \tau'_2, \tau'_3, \tau'_{(2,1,0)}, \tau'_{(1,2,0)}, \tau'_{(2,0,1)}, \tau'_{(1,0,2)}, \tau'_{(0,2,1)}, \tau'_{(0,1,2)},$$

and the rows represent the component $\tilde{Y}_{(i_1,i_2,i_3)}$ in the following order

$$(3,0,0), (0,3,0), (0,0,3), (2,1,0), (1,2,0), (2,0,1), (1,0,2), (0,2,1), (0,1,2), (1,1,1).$$

For $\gamma \in \mathrm{CH}_1(\tilde{Y})$ of the above form the maps $\Phi_{(i_1,i_2,i_3)} := \Phi_{\tilde{\mathfrak{X}},\tilde{Y}_{(i_1,i_2,i_3)}}$ are given by

$$\begin{aligned}
\Phi_{(3,0,0)}(\gamma) &= -\gamma_1|_{(Y_1\cap Y_2)_L} - \gamma_1|_{(Y_1\cap Y_3)_L} + \alpha_{(2,1,0)} + \alpha_{(2,0,1)} + \alpha'_1 && \in \mathrm{CH}_0(\tilde{Y}_{(3,0,0)}),\\
\Phi_{(0,3,0)}(\gamma) &= -\gamma_2|_{(Y_1\cap Y_2)_L} - \gamma_2|_{(Y_2\cap Y_3)_L} + \alpha_{(1,2,0)} + \alpha_{(0,2,1)} + \alpha'_2 && \in \mathrm{CH}_0(\tilde{Y}_{(0,3,0)}),\\
\Phi_{(0,0,3)}(\gamma) &= -\gamma_3|_{(Y_1\cap Y_3)_L} - \gamma_3|_{(Y_2\cap Y_3)_L} + \alpha_{(1,0,2)} + \alpha_{(0,1,2)} + \alpha'_3 && \in \mathrm{CH}_0(\tilde{Y}_{(0,0,3)}),\\
\Phi_{(2,1,0)}(\gamma) &= \gamma_1|_{(Y_1\cap Y_2)_L} - 2\alpha_{(2,1,0)} + \alpha_{(1,2,0)} - \alpha'_1 - \alpha'_{(2,1,0)} + \alpha'_{(1,2,0)} + \alpha'_{(2,0,1)} && \in \mathrm{CH}_0(\tilde{Y}_{(2,1,0)}),\\
\Phi_{(1,2,0)}(\gamma) &= \gamma_2|_{(Y_1\cap Y_2)_L} + \alpha_{(2,1,0)} - 2\alpha_{(1,2,0)} - \alpha'_2 + \alpha'_{(2,1,0)} - \alpha'_{(1,2,0)} + \alpha'_{(0,2,1)} && \in \mathrm{CH}_0(\tilde{Y}_{(1,2,0)}),\\
\Phi_{(2,0,1)}(\gamma) &= \gamma_1|_{(Y_1\cap Y_3)_L} - 2\alpha_{(2,0,1)} + \alpha_{(1,0,2)} - \alpha'_1 + \alpha'_{(2,1,0)} - \alpha'_{(2,0,1)} + \alpha'_{(1,0,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(2,0,1)}),\\
\Phi_{(1,0,2)}(\gamma) &= \gamma_3|_{(Y_1\cap Y_3)_L} + \alpha_{(2,0,1)} - 2\alpha_{(1,0,2)} - \alpha'_3 + \alpha'_{(2,0,1)} - \alpha'_{(1,0,2)} + \alpha'_{(0,1,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(1,0,2)}),\\
\Phi_{(0,2,1)}(\gamma) &= \gamma_2|_{(Y_2\cap Y_3)_L} - 2\alpha_{(0,2,1)} + \alpha_{(0,1,2)} - \alpha'_2 + \alpha'_{(1,2,0)} - \alpha'_{(0,2,1)} + \alpha'_{(0,1,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,2,1)}),\\
\Phi_{(0,1,2)}(\gamma) &= \gamma_3|_{(Y_2\cap Y_3)_L} + \alpha_{(0,2,1)} - 2\alpha_{(0,1,2)} - \alpha'_3 + \alpha'_{(1,0,2)} + \alpha'_{(0,2,1)} - \alpha'_{(0,1,2)} && \in \mathrm{CH}_0(\tilde{Y}_{(0,1,2)}),\\
\Phi_{(1,1,1)}(\gamma) &= \alpha'_1 + \alpha'_2 + \alpha'_3 - \sum_{\sigma\in S_3} \alpha'_{\sigma(2,1,0)} && \in \mathrm{CH}_0(\tilde{Y}_{(1,1,1)}).
\end{aligned}$$

Note that $\mathrm{CH}_0(\tilde{Y}_{(2,1,0)}) = \mathrm{CH}_0((Y_1\cap Y_2)_L)$ as $\tilde{Y}_{(2,1,0)}$ is a blow-up in smooth centers of a locally trivial $\mathbb{P}^1$-bundle over $(Y_1\cap Y_2)_L$ (and similarly for the other components), i.e. we view the zero-cyles $\alpha_{(1,2,0)}$ as a zero-cycle on $\tilde{Y}_{(2,1,0)}$. The key lemma reads

$$\begin{aligned}
\sum_{\substack{i_1,i_2,i_3\\ i_1>0}} i_1 q_*\Phi_{(i_1,i_2,i_3)}(\gamma) &= -(q_*\gamma_1)|_{Y_1\cap Y_2} - (q_*\gamma_1)|_{Y_1\cap Y_3} + (q_*\gamma_2)|_{Y_1\cap Y_2} + (q_*\gamma_3)|_{Y_1\cap Y_3} = \Phi_{\mathfrak{X},Y_1}(q_*\gamma),\\
\sum_{\substack{i_1,i_2,i_3\\ i_2>0}} i_2 q_*\Phi_{(i_1,i_2,i_3)}(\gamma) &= -(q_*\gamma_2)|_{Y_1\cap Y_2} - (q_*\gamma_2)|_{Y_2\cap Y_3} + (q_*\gamma_1)|_{Y_1\cap Y_2} + (q_*\gamma_3)|_{Y_2\cap Y_3} = \Phi_{\mathfrak{X},Y_2}(q_*\gamma),\\
\sum_{\substack{i_1,i_2,i_3\\ i_3>0}} i_3 q_*\Phi_{(i_1,i_2,i_3)}(\gamma) &= -(q_*\gamma_3)|_{Y_1\cap Y_3} - (q_*\gamma_3)|_{Y_2\cap Y_3} + (q_*\gamma_1)|_{Y_1\cap Y_3} + (q_*\gamma_2)|_{Y_2\cap Y_3} = \Phi_{\mathfrak{X},Y_3}(q_*\gamma),
\end{aligned}$$

which can be checked immediately by the above description using the observations $q_*\gamma = q_*\gamma_1 + q_*\gamma_2 + q_*\gamma_3$ and $(q_*\gamma_i)|_{Y_i\cap Y_j} = q_*\left(\gamma_i|_{(Y_i\cap Y_j)_L}\right)$ by [Ful98, Prop. 2.3 (c)].

# 4. Irrationality of low degree hypersurfaces

## 4.1. Introduction

A variety $X$ over a field $k$ is retract rational if there is some integer $N \geq \dim X$ and rational maps $f\colon X \dashrightarrow \mathbb{P}^N$ and $g\colon \mathbb{P}^N \dashrightarrow X$ such that the composition $g \circ f$ is defined and agrees with $\mathrm{id}_X$. This notion is a direct analogue of retracts in topology; it was introduced into birational geometry by Saltman [Sal82, Sal84] in the 1980s. Equivalently, $X$ is retract rational if there are an integer $N \geq \dim X$, open dense subsets $U \subset X$ and $V \subset \mathbb{P}^N$, and a morphism $V \to U$ which is surjective on $L$-rational points for all field extensions $L/k$—this closely relates the notion to the classical question of parametrizing solutions of polynomial equations by rational functions. Rational or stably rational varieties are retract rational.

By the work of Asok–Morel [AM11, Theorem 2.3.6] and Kahn–Sujatha [KS15, Theorem 8.5.1 and Proposition 8.6.2], a smooth proper retract rational variety $X$ is $\mathbb{A}^1$-connected in the sense of $\mathbb{A}^1$-homotopy theory, i.e. $\pi_0^{\mathbb{A}^1}(X) = \{*\}$. By [AM11, Remark 2.4.8], $\mathbb{A}^1$-connectedness is equivalent to separable $R$-triviality, which means that $X(L)/R = \{*\}$ for any separable field extension $L/k$, where $R$ denotes the equivalence relation on $X(L)$ generated by $x \sim y$ whenever $x, y \in X(L)$ lie on the same rational curve (defined over $L$). In other words, $\mathbb{A}^1$-connectedness provides an arithmetic analogue of rational chain connectedness, which requires that any two $L$-rational points can be connected by a chain of rational curves for any algebraically closed field extension $L$ of $k$. The following table illustrates the known implications for smooth proper varieties:

$$\begin{tikzcd}
\text{rational} \arrow[r, Rightarrow] & \text{stably rational} \arrow[r, Rightarrow] & \text{retract rational} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathbb{A}^1\text{-connected} \arrow[d, Rightarrow] \\
 & & \text{unirational} \arrow[r, Rightarrow] & \text{rationally chain connected}
\end{tikzcd}$$

The rationality problem for a given rationally connected variety $X$ asks 'how rational it is', that is, which of the properties in the above diagram are satisfied.

Not every unirational variety is $\mathbb{A}^1$-connected [AM72] and not every stably rational variety is rational [BCTSS85]. Moreover, there are retract rational varieties (over non-closed fields) that are not stably rational, see e.g. [EM75, Theorem 1.5 and Theorem 2.3]; it is an open problem to produce such examples over algebraically closed fields. Whether any $\mathbb{A}^1$-connected variety is retract rational is open over any field.

A smooth proper $\mathbb{A}^1$-connected variety with a $k$-rational point has universally trivial Chow group of zero-cycles and hence admits a decomposition of the diagonal [BS83], which is an interesting motivic and cycle-theoretic property in itself.

### 4.1.1. Hypersurfaces

A particularly interesting class of varieties for the rationality problem are smooth projective hypersurfaces $X \subset \mathbb{P}_k^{N+1}$ of degree $d$ and dimension $N$ over a field $k$. The interesting range for the problem is when $d \leq N+1$, in which case $X$ is Fano and thus rationally chain connected by [Cam92, KMM92].

If $2^{d!} \leq N+1$ and $k = \mathbb{C}$, then $X$ is unirational, see [BR21, HMP98]. If $d = N+1$ and $k = \mathbb{C}$, then $X$ is irrational (in fact birationally rigid) by a theorem of de Fernex [deF13, deF16], which extends earlier results by Iskovskikh–Manin [IM72] and Pukhlikov [Pu87, Pu98]. If $k = \mathbb{C}$ and $X \subset \mathbb{P}_{\mathbb{C}}^{N+1}$ is very general of degree $d \geq 2\lceil \frac{N+3}{3} \rceil$, then it is not ruled and hence not rational by a theorem of Kollár [Kol95]. Under the slightly weaker bound $d \geq 2\lceil \frac{N+2}{3} \rceil$, Totaro [Tot16a] showed that such hypersurfaces do not admit a decomposition of the diagonal, hence are neither stably nor retract rational, nor $\mathbb{A}^1$-connected. This used [Voi15, CTP16]. Totaro's result was improved in [Sch19b], where the same result under the logarithmic bound $d \geq \log_2 N + 2$, $N \geq 3$ and over fields of characteristic $\neq 2$ was proven; a similar bound holds in characteristic 2 by [Sch21a].

The logarithmic degree bound in [Sch19b] is equivalent to $N \leq 2^{d-2}$. In the case of stable rationality over fields of characteristic zero, Moe [Moe23] used the methods from [NS19, KT19, NO22] to improve this logarithmic bound by a factor $\frac{(d+1)}{4}$ to cover the cases $N \leq (d+1)2^{d-4}$. This chapter generalizes Moe's result as follows:

**Theorem 4.1.1.** *Let $k$ be a field of characteristic different from 2. Then a very general hypersurface $X \subset \mathbb{P}_k^{N+1}$ of degree $d \geq 4$ and dimension $N \leq (d+1)2^{d-4}$ does not admit a decomposition of the diagonal, hence is neither stably nor retract rational, nor $\mathbb{A}^1$-connected.*

While stable irrationality in characteristic zero follows in the above degree range from [Moe23, Theorem 5.2], the assertion on retract rationality and $\mathbb{A}^1$-connectedness are new. In positive characteristic, for all $N \leq (d+1)2^{d-4}$ not covered by [Sch19b], even rationality was previously open. For fixed degree $d$, a proportion of roughly $\frac{d-3}{d+1}$ cases are new. The first new case concerns quintics of dimension $N = 10$.

By a very general hypersurface $X$ over a field $k$ we mean one where the coefficients of a defining equation are algebraically independent over the prime field, see Remark 2.4.5. With this definition, very general hypersurfaces exist over any field (not necessarily uncountable) of sufficiently large transcendence degree over the prime field, cf. Lemma 2.4.3.

In characteristic 2, we obtain an analogous result under a slightly weaker bound:

**Theorem 4.1.2.** *Let $k$ be a field of characteristic 2. Then a very general hypersurface $X \subset \mathbb{P}^{N+1}_k$ of degree $d \geq 5$ and dimension $N \leq \frac{d+1}{3}2^{d-4}$ does not admit a decomposition of the diagonal, hence is neither stably nor retract rational, nor $\mathbb{A}^1$-connected.*

In characteristic 2, the logarithmic bound in [Sch21a] is given by $N \leq 2^{d-3}$. The above theorem improves this by a factor $(d+1)/6$; for fixed $d$, the proportion of new cases is given by $(d-5)/(d+1)$.

Slightly better numerical bounds than in Theorems 4.1.1 and 4.1.2 can be extracted from Theorem 4.6.1 (see also Theorem 4.1.3) below, which is our main result.

Our arguments allow us to bound the torsion order $\mathrm{Tor}(X)$ of the above hypersurfaces $X \subset \mathbb{P}^{N+1}_k$, i.e. the smallest positive integer $e$ such that $e \cdot \Delta_X$ decomposes in the Chow group of $X \times X$ (or $e = \infty$ if no such integer exists). If $\mathrm{Tor}(X) > 1$, then $X$ does not admit a decomposition of the diagonal, hence is not $\mathbb{A}^1$-connected. Moreover, any dominant generically finite map $f\colon \mathbb{P}^{\dim X} \dashrightarrow X$ has degree $\deg f$ divisible by $\mathrm{Tor}(X)$ and so the torsion order yields an interesting lower bound on the possible degrees of unirational parametrizations of $X$.

If $X \subset \mathbb{P}^{N+1}_k$ is a smooth Fano hypersurface of degree $d$ over some field $k$, then $\mathrm{Tor}(X)$ always divides $d!$, see [Roi72] and [CL17, Proposition 5.2]. This yields an upper bound for the possible torsion orders of Fano hypersurfaces. We then have the following result, which improves the previously known lower bounds from [CL17, Sch21a].

**Theorem 4.1.3.** *Let $k$ be a field and let $m \geq 2$ be an integer invertible in $k$. Let $n \geq 2$, $r \leq 2^n - 2$, and $s \leq \left(\lfloor \frac{n}{m} \rfloor - 1\right)\left(2^{n-1} - 1\right)$ be non-negative integers and* *write $N := n + r + s$. Then the torsion order of a very general Fano hypersurface $X \subset \mathbb{P}_k^{N+1}$ of degree $d \geq m + n$ is divisible by $m$.*

In Theorem 4.6.1 below we prove the above result under the weaker upper bound on $s$ given by $s \leq \sum_{l=1}^{n} \binom{n}{l} \left\lfloor \frac{n-l}{m} \right\rfloor$. Previously, the best known bound on the torsion orders of hypersurfaces was contained in [Sch21a] and corresponds to the case $s = 0$.

### 4.1.2. Outline of the argument

This paper provides a flexible cycle-theoretic analogue of the motivic obstruction from [NS19, KT19], which applies to degenerations into unions of varieties such that the obstruction lies in some lower-dimensional strata, and not in the components, as in [Voi15, CTP16, Sch19a].

Previously, a solution to this problem has been proposed by the second named author with Pavic in [PS23], with an important recent generalization by the first named author in [Lan24]. The main weakness of our previous approach is the fact that one has to compute an explicit strictly semi-stable model of the degeneration in question, control the combinatorics of the dual complex of the special fibre and control the Chow groups of 0- and 1-cycles of all components, which itself is a notoriously difficult task for almost any given smooth projective variety. These tasks have been solved in a computationally involved manner for quartic fivefolds [PS23] and $(3,3)$-complete intersections in $\mathbb{P}^7$ [LS23]. However, we do not see how to apply our obstruction from [PS23, Lan24] systematically to examples of higher dimensions, such as to the higher-dimensional complete intersections or hypersurfaces treated via the aforementioned motivic method in [NO22, Moe23]; the total space of these degenerations are not strictly semi-stable (they have toric singularities) and the special fibre has a large number of components whose Chow groups seem inaccessible.

The main improvement proposed in this paper is an extension of our previous method from [PS23, Lan24] to the non-proper case and hence in effect to (very) singular degenerations, that we could not handle before. On a technical level, the idea is to work with pairs of a variety $X$ and a closed subset $W \subset X$. To state our obstruction, we say that a variety $X$ admits a decomposition of the diagonal with respect to a closed subset $W \subset X$, if the diagonal point $\delta_X \in \mathrm{CH}_0(X_{k(X)})$ lies in the image of $\mathrm{CH}_0(W_{k(X)}) \to \mathrm{CH}_0(X_{k(X)})$, see Section 4.2 below. With this terminology, an ordinary decomposition of the diagonal corresponds to one with respect to a zero-dimensional closed subset.

The obstruction to rationality that we introduce and exploit reads as follows.

**Theorem 4.1.4.** *Let $R$ be a discrete valuation ring with algebraically closed residue field $k$ and fraction field $K$. Let $\mathcal{X} \to \operatorname{Spec} R$ be a proper flat $R$-scheme with geometrically integral generic fibre $X = \mathcal{X} \times_R K$ and special fibre $Y = \mathcal{X} \times_R k$. Let $W_{\mathcal{X}} \subset \mathcal{X}$ be a closed subscheme and let $W_X \coloneqq W_{\mathcal{X}} \cap X$ and $W_Y \coloneqq W_{\mathcal{X}} \cap Y$ be the intersections with the generic fibre and the special fibre, respectively. Assume that the following conditions are satisfied:*

*(1) $\mathcal{X}^\circ \coloneqq \mathcal{X} \setminus W_{\mathcal{X}}$ is strictly semi-stable over $R$ (see Definition 2.3.2);*

*(2) $Y^\circ \coloneqq Y \setminus W_Y$ consists of two components $Y_0^\circ, Y_1^\circ$, with intersection $Z^\circ = Y_0^\circ \cap Y_1^\circ$.*

*If the geometric generic fibre $\bar{X} = X \times \bar{K}$ admits a decomposition of the diagonal relative to the closed subset $W_{\bar{X}} \coloneqq W_X \times_K \bar{K}$, then, for any field extension $L/k$, the map*

$$\Psi_{Y_L^\circ}\colon \mathrm{CH}_1(Y_0^\circ \times_k L) \oplus \mathrm{CH}_1(Y_1^\circ \times_k L) \longrightarrow \mathrm{CH}_0(Z^\circ \times_k L), \quad (\gamma_0, \gamma_1) \mapsto \gamma_0|_{Z^\circ} - \gamma_1|_{Z^\circ}$$

*is surjective modulo any integer $m$ that is invertible in $k$, where $\gamma_i|_{Z^\circ}$ denotes the pullback of $\gamma_i$ along the regular embedding $Z^\circ \hookrightarrow Y_i^\circ$, see [Ful98, Remark 2.3].*

Theorem 4.1.4 admits a generalization to the case where $Y^\circ$ is an snc scheme without triple intersections, see Theorem 4.3.3 below. Our arguments do not generalize to degenerations where the obstruction lies in deeper strata, see Remark 4.3.15 below.

The obstruction map $\Psi_{Y_L^\circ}$ from Theorem 4.1.4 is a refined version of the one in [PS23, Lan24]. The presence of $W_{\mathcal{X}}$ in the above theorem yields the extra flexibility that was missing in [PS23, Lan24]. In applications we will declare the singular locus of $\mathcal{X}$, the non-snc locus of $Y$, as well as all but two components of $Y$, to be contained in $W_{\mathcal{X}}$. In particular, the family $\mathcal{X}$ in the above theorem may be quite singular and the combinatorics of the special fibre $Y$ can be complicated.

To explain the mechanism of the above theorem, assume that $Z^\circ$ is integral and let $Z \subset Y$ be the closure of $Z^\circ$. Assume further that $\mathrm{CH}_1(Y_i^\circ \times_k L) = 0$ for $i = 0, 1$ and any field extension $L/k$. (This will not hold on the nose in practice, but we will be able to achieve this after degeneration and show that this suffices for the argument.) In particular, the map $\Psi_{Y_L^\circ}$ in the above theorem is the zero map for every field extension $L/k$. Applying this to the function field $L = k(Z)$ of $Z$ and assuming that $\bar{X}$ admits a decomposition of the diagonal with respect to $W_{\bar{X}}$, we get that the image of the diagonal point $\delta_Z$ in $\mathrm{CH}_0(Z^\circ \times k(Z))$ vanishes, and so, by the localization sequence, $Z$ admits a decomposition with respect to $Z \cap W_Y$. In other words, $\bar{X}$ admits no decomposition of the diagonal with respect to $W_{\bar{X}}$ as long as $Z$ admits no decomposition with respect to $Z \cap W_Y$. Examples of this strategy are illustrated in Examples 4.3.5 and 4.3.6 below. A simplified version of Theorem 4.1.4 is stated in Theorem 4.3.8.

Since $\dim Z = \dim X - 1$, the above reasoning sets the stage for an inductive argument where one increases the dimension by one in each step. What makes this work is the observation that the examples of Fano hypersurfaces without a decomposition of the diagonal in [Sch19b, Sch21a] can in fact be shown to have no decomposition of the diagonal with respect to a large class of divisors, see Theorem 4.5.1 below. This will serve as the start of our induction. For the induction step we degenerate a given hypersurface of degree $d$ to a union of two rational varieties which meet along the lower-dimensional hypersurface of degree $d$ that we have produced in the previous step of the induction, see Section 4.4 below for the precise degeneration we pick. This step is inspired by [Moe23]. The total space of our degeneration as well as the fibres and their components will be (very) singular. The singularities are not toric and so even in characteristic zero, the method in [KT19, NS19, NO22] does not seem to apply to our degeneration.

**Remark 4.1.5.** After completion of the paper [LS25], James Hotchkiss and David Stapleton informed us that they have independently obtained a different argument which shows that, over fields of characteristic zero, the hypersurfaces in Theorem 4.1.1 are not $\mathbb{A}^1$-connected and hence not retract rational, see [HS25]. Their approach relies on a homotopy-theoretical lift of the obstruction from [NS19, KT19]. As in [NS19, KT19, NO22, Moe23], the assumption on the characteristic is needed to be able to apply weak factorization.

## 4.2. Torsion orders and decompositions of the diagonal relative to a closed subset

Let $X$ be a variety over a field $k$. We say that $e$ times the diagonal of $X$ decomposes if there is a zero-cycle $z \in \mathrm{CH}_0(X)$ such that

$$e \cdot \Delta_X = z \times X + Z \in \mathrm{CH}_{\dim X}(X \times_k X),$$

for some cycle $Z$ whose support does not dominate the second factor of $X \times_k X$, see [BS83]. The torsion order of $X$, denoted by $\mathrm{Tor}(X)$, is the smallest positive integer

$e$ such that a decomposition as above exists; it is $\infty$ if no such integer exists, see e.g. [CL17, Kah17, Sch21a]. We say that $X$ admits a decomposition of the diagonal if $\mathrm{Tor}(X) = 1$. By the localization exact sequence [Ful98, §1.8], this is equivalent to saying that $\delta_X \in \mathrm{im}(\mathrm{CH}_0(W_{k(X)}) \to \mathrm{CH}_0(X_{k(X)}))$ for a zero-dimensional closed subset $W \subset X$, where $\delta_X \in X_{k(X)}$ denotes the point induced by the diagonal $\Delta_X \subset X \times_k X$. This leads to the following simple but useful variant.

**Definition 4.2.1.** Let $X$ be a variety over a field $k$ and let $\Lambda$ be a ring. We say that $X$ admits a $\Lambda$-decomposition of the diagonal relative to a closed subset $W \subset X$ if

$$\delta_X \in \mathrm{im}\left(\mathrm{CH}_0(W_{k(X)}, \Lambda) \to \mathrm{CH}_0(X_{k(X)}, \Lambda)\right),$$

where $\delta_X$ denotes the diagonal point induced by the diagonal $\Delta_X \subset X \times_k X$. If $\Lambda = \mathbb{Z}$, we also say that $X$ admits a decomposition (or integral decomposition) of the diagonal relative to $W$.

Variants of the notion of a decomposition of diagonal relative to a closed subscheme appear for example in [BS83], [Voi17, Definition 1.2], and [CL17, Definition 1.1].

**Remark 4.2.2.** By the above discussion, $X$ admits a decomposition of the diagonal if and only if it admits a decomposition relative to a closed subset of dimension zero.

By the localization sequence, the condition on $\delta_X$ in Definition 4.2.1 is equivalent to

$$\delta_X \in \ker\left(\mathrm{CH}_0(X_{k(X)}, \Lambda) \to \mathrm{CH}_0(U_{k(X)}, \Lambda)\right),$$

where $U = X \setminus W$. It follows that a $\Lambda$-decomposition of the diagonal relative to $W \subset X$ is the same thing as a $\Lambda$-decomposition of the diagonal of $U$, relative to the empty set. These observations lead us to the following relative version of the aforementioned torsion order studied for instance in [CL17, Kah17, Sch21a].

**Definition 4.2.3.** Let $X$ be a variety over a field $k$ and let $\Lambda$ be a ring. The $\Lambda$-torsion order of $X$ relative to a closed subset $W \subset X$, denoted by $\mathrm{Tor}^{\Lambda}(X, W)$, is the order of the element

$$\delta_X|_U = \delta_U \in \mathrm{CH}_0(U_{k(X)}, \Lambda),$$

where $U = X \setminus W$.

By definition, $\mathrm{Tor}^{\Lambda}(X, W) \in \mathbb{N} \cup \{\infty\}$. If $\Lambda$ has characteristic $c \neq 0$, then $\mathrm{Tor}^{\Lambda}(X, W)$ divides $c$ and hence is finite.

We remark that the torsion order of proper varieties relative to the empty set has somewhat pathological behaviour. For instance, the $\mathbb{Z}$-torsion order of a proper variety with respect to the empty set is always $\infty$, but it may or may not be finite relative to a non-empty closed subset $W \subset X$.

**Remark 4.2.4.** The $\Lambda$-torsion order of $X$ relative to $W$ is 1 if and only if $X$ admits a $\Lambda$-decomposition of the diagonal relative to $W$. The $\Lambda$-torsion order of $X$ relative to $W$ is nothing but the $\Lambda$-torsion order of $X \setminus W$, relative to the empty set.

**Remark 4.2.5.** If $X$ is an algebraic scheme over $k$ (not necessarily irreducible) and $W \subset X$ is a closed subset such that $U = X \setminus W$ is integral, then we can, in view of Remark 4.2.4, still define $\mathrm{Tor}^{\Lambda}(X, W)$ via the order of the element

$$\delta_U \in \mathrm{CH}_0(U_{k(U)}, \Lambda).$$

**Lemma 4.2.6.** *Let $X$ be a variety over a field $k$ and let $W \subset X$ be a closed subset. Then the following hold:*

*(a) For all $m \in \mathbb{Z}$, $\mathrm{Tor}^{\mathbb{Z}/m}(X, W) \mid \mathrm{Tor}^{\mathbb{Z}}(X, W)$.*

*(b) Let $W' \subset W \subset X$ be a closed subset, then $\mathrm{Tor}^{\Lambda}(X, W) \mid \mathrm{Tor}^{\Lambda}(X, W')$.*

*(c) $\mathrm{Tor}(X)$ is the minimum of the relative torsion orders $\mathrm{Tor}^{\mathbb{Z}}(X, W)$ where $W \subset X$ runs through all closed subsets of dimension zero.*

*(d) If $X$ is proper and $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ is an isomorphism, then $\mathrm{Tor}(X) = \mathrm{Tor}^{\mathbb{Z}}(X, W)$ for any closed subset $W \subset X$ of dimension zero, which contains a zero-cycle of degree 1.*

*(e) If $k = \bar{k}$ is algebraically closed, then $\mathrm{Tor}^{\Lambda}(X, W) = \mathrm{Tor}^{\Lambda}(X_L, W_L)$ for any ring $\Lambda$ and any field extension $L/k$.*

*Proof.* Items (a)–(d) follow easily from the definitions and the above discussions. To prove item (e), note that $\mathrm{Tor}^{\Lambda}(X_L, W_L) \mid \mathrm{Tor}^{\Lambda}(X, W)$ (even without asking that $k$ is algebraically closed). The converse divisibility statement follows via a straightforward "spreading out and specialization at a $k$-point" argument. This concludes the proof of the lemma. $\square$

**Lemma 4.2.7.** *Let $X$ be a smooth projective variety over an algebraically closed field $k$. If $H^0(X, \Omega_X^1) = 0$, then the torsion order $\mathrm{Tor}(X) = \infty$ or $\mathrm{CH}_0(X) \simeq \mathbb{Z}$.*

The lemma is a consequence of Rojtman's theorem and says that the torsion order of smooth projective varieties over an algebraically closed field with $h^{1,0} = 0$ is either infinite or the torsion order relative to any non-empty closed zero-dimensional subset.

*Proof.* Let $X$ be a smooth projective variety over an algebraically closed field with Hodge number $h^{1,0} = 0$. Assume that the torsion order $\mathrm{Tor}(X)$ is finite. Then it follows by an "action of correspondence" argument that the group of degree 0 zero-cycles $A_0(X)$ is $l := \mathrm{Tor}(X)$-torsion. By Rojtman's theorem [Blo79, Roi80, Mil82], the Albanese morphism induces an isomorphism

$$A_0(X) \longrightarrow \mathrm{Alb}(X)(l)$$

between $\mathrm{CH}_0(X)_0$ and the $l$-torsion points of the Albanese variety of $X$. By [Igu55], we have $\dim \mathrm{Alb}(X) \le h^{1,0}(X) = 0$. Hence $A_0(X) = 0$, which proves the lemma. □

The next lemma explains the geometric meaning of $\Lambda$-torsion orders.

**Lemma 4.2.8.** *Let $X$ be a proper variety over a field $k$ and let $\Lambda$ be a ring. Assume that $X$ admits a resolution of singularities or that the exponential characteristic of $k$ is invertible in $\Lambda$. If for some closed subset $W \subset X$ the complement $U := X \setminus W$ is smooth, then $\mathrm{CH}_0(U_L, \Lambda)$ is $\mathrm{Tor}^\Lambda(X, W)$-torsion for all field extensions $L/k$.*

*Proof.* Since $\mathrm{Tor}^\Lambda(X_L, W_L)$ divides $\mathrm{Tor}^\Lambda(X, W)$, we can assume without loss of generality that $L = k$. By work of Temkin [Tem17], we can pick an alteration $\tau : X' \to X$ whose degree is a power of the exponential characteristic. Moreover, we can choose $\tau$ to be of degree 1 if $X$ admits a resolution of singularities. We then let $W' := \tau^{-1}(W)$ and $U' = X' \setminus W'$. We further let $V \subset U$ be the locus over which $\tau$ is étale and define $V' := \tau^{-1}(V)$.

Let $m := \mathrm{Tor}^\Lambda(X, W)$. By assumptions, $m \cdot \delta_X \in \mathrm{CH}_0(X_{k(X)}, \Lambda)$ vanishes when restricted to $U$. Hence,

$$m \cdot \tau^* \delta_U = 0 \in \mathrm{CH}_0(U'_{k(U)}, \Lambda).$$

The base change of this class to the field extension $k(U')$ of $k(U)$ still vanishes. If we spread this out and use the localization sequence, we find that

$$m \cdot \Gamma = [Z_1] + [Z_2] \in \mathrm{CH}_{\dim X'}(X' \times X', \Lambda), \tag{4.2.1}$$

where $\Gamma \subset X' \times X'$ denotes the closure of the locus

$$(\tau|_{V'} \times \tau|_{V'})^{-1}(\Delta_V) = \{(x,y) \in V' \times V' \mid \tau(x) = \tau(y) \in V\} \subset X' \times X',$$

and $Z_1$, $Z_2$ denote some cycles with

$$\operatorname{supp} Z_1 \subset W' \times X' \quad \text{and} \quad \operatorname{supp} Z_2 \subset X' \times D$$

for some nowhere dense closed subset $D \subsetneq X'$.

Let now $z \in \mathrm{CH}_0(U,\Lambda)$. By Chow's moving lemma, we can assume that $\operatorname{supp} z \subset V$ and $\operatorname{supp} z \cap \tau(D) = \emptyset$. We aim to show $m \cdot z = 0 \in \mathrm{CH}_0(U,\Lambda)$. To this end, let $p : X' \times X' \to X'$ and $q : X' \times X' \to X'$ denote the projections to the first and second factors, respectively. Since $X'$ is smooth and proper over $k$, we can define pullbacks along correspondences in $\mathrm{CH}^{\dim X}(X' \times X', \Lambda)$.

By assumption, the support of $z$ lies in the locus over which $\tau$ is étale: $\operatorname{supp} z \subset V$. We can thus define the zero-cycle $\tau^* z$ on $X'$ on the level of cycles via the preimages of the points in the support of $z$. We then apply the correspondence $\Gamma$ and get

$$m \cdot \Gamma^*(\tau^* z) = p_*(m\Gamma \cdot q^* \tau^* z) \in \mathrm{CH}_0(X', \Lambda).$$

Since the support of $z$ is contained in the locus $V$ over which $\tau$ is étale, a direct computation shows that the above zero-cycle has the property that

$$\tau_*(m \cdot \Gamma^*(\tau^* z)) = (\deg \tau)^2 \cdot m \cdot z \in \mathrm{CH}_0(X, \Lambda). \tag{4.2.2}$$

Conversely, by (4.2.1), we know that this cycle is rationally equivalent to

$$\tau_*(\Gamma^*(m \cdot \tau^* z)) = \tau_*([Z_1]^*(\tau^* z) + [Z_2]^*(\tau^* z)) \in \mathrm{CH}_0(X, \Lambda).$$

Since $\operatorname{supp} Z_2 \subset X' \times D$ and $\operatorname{supp} z \cap \tau(D) = \emptyset$, we have $[Z_2]^*(\tau^* z) = 0$ and so

$$\Gamma^*(m \cdot \tau^* z) = [Z_1]^*(\tau^* z) \in \mathrm{CH}_0(X', \Lambda).$$

Since $\operatorname{supp} Z_1 \subset W' \times X'$, the above class lies in the image of $\mathrm{CH}_0(W', \Lambda) \to \mathrm{CH}_0(X', \Lambda)$. Hence,

$$\tau_* \Gamma^*(m \cdot \tau^* z) \in \operatorname{im}(\mathrm{CH}_0(W, \Lambda) \to \mathrm{CH}_0(X, \Lambda)).$$

By (4.2.2), it follows that

$$(\deg \tau)^2 \cdot m \cdot z \in \operatorname{im}(\mathrm{CH}_0(W, \Lambda) \longrightarrow \mathrm{CH}_0(X, \Lambda)).$$

By the localization sequence, this implies

$$\deg(\tau)^2 \cdot m \cdot z|_U = 0 \in \mathrm{CH}_0(U, \Lambda).$$

Since $\deg(\tau)^2$ is either one or a power of the exponential characteristic of $k$, it is invertible in $\Lambda$ by assumption. We deduce, as we want, that $z \in \mathrm{CH}_0(U, \Lambda)$ is $m$-torsion. This concludes the proof of the lemma. □

We will use that $\Lambda$-torsion orders relative to closed subsets are well-behaved under specialization, as shown by the following lemma.

**Lemma 4.2.9.** *Let $R$ be a discrete valuation ring with fraction field $K$ and residue field $k$. Let $\mathcal{X} \to \operatorname{Spec} R$ be a separated flat $R$-scheme of finite type with generic fibre $X = \mathcal{X} \times K$ and special fibre $Y = \mathcal{X} \times k$. Let $\bar{X} = X \times \bar{K}$ and $\bar{Y} = Y \times \bar{k}$ be the base changes to algebraic closures of $K$ and $k$, respectively. Let $W_{\mathcal{X}} \subset \mathcal{X}$ be a closed subset with $W_{\bar{X}} := W_{\mathcal{X}} \times \bar{K} \subset \bar{X}$ and $W_{\bar{Y}} := W_{\mathcal{X}} \times \bar{k} \subset \bar{Y}$. Assume that the fibres of $U = \mathcal{X} \setminus W_{\mathcal{X}}$ over $R$ are non-empty and geometrically integral.*

*Then we have*

$$\mathrm{Tor}^{\Lambda}(\bar{Y}, W_{\bar{Y}}) \mid \mathrm{Tor}^{\Lambda}(\bar{X}, W_{\bar{X}}).$$

*Proof.* By inflation of local rings [Bou06, Chapter IX, Appendice §2, Corollaire du Théorème 1 and Exercice 4], there is an unramified extension of discrete valuation rings $R'/R$ such that the residue field of $R'$ is $\bar{k}$. Up to a base change along $R'/R$ we can thus assume that $k$ is algebraically closed. (This uses item (e) in Lemma 4.2.6.)

Let $m := \mathrm{Tor}^{\Lambda}(\bar{X}, W_{\bar{X}})$. Then there is a finite extension $K'/K$ such that the $\Lambda$-torsion order of $X \times K'$ relative to $W_X \times K'$ is $m$. Let $R_{K'} \subset K'$ be the integral closure of $R$ in $K'$ and let $R' \subset K'$ be the localization of $R_{K'}$ at a maximal ideal lying over the maximal ideal of $R$. Then $R'$ is a discrete valuation ring with $\operatorname{Frac} R' = K'$ and $R \subset R'$. Up to a base change along $\operatorname{Spec} R' \to \operatorname{Spec} R$, we can then assume that $K = K'$. Hence, $m = \mathrm{Tor}^{\Lambda}(X, W_X)$ and it remains to show that

$$\mathrm{Tor}^{\Lambda}(Y, W_Y) \mid m. \tag{4.2.3}$$

Let $U = \mathcal{X} \setminus W_{\mathcal{X}}$ with generic fibre $U_\eta := U \times K$ and special fibre $U_0 := U \times k$. By assumptions, $U \to \operatorname{Spec} R$ is flat with non-empty geometrically integral fibres.

Let $A$ be the local ring of $U$ at the generic point of the special fibre $U_0$. Since $U_0$ is integral, $A$ is a discrete valuation ring with residue field $k(U_0)$ and fraction field $K(U_\eta)$. We then apply Fulton's specialization map on Chow groups to the base change $U_A \coloneqq U \times_R A$ and obtain a group homomorphism

$$\mathrm{sp} : \mathrm{CH}_0(U_\eta \times K(U_\eta), \Lambda) \longrightarrow \mathrm{CH}_0(U_0 \times k(U_0), \Lambda)$$

with $\mathrm{sp}(\delta_{U_\eta}) = \delta_{U_0}$, see Lemma 2.2.1. This implies (4.2.3), because $m = \mathrm{Tor}^\Lambda(X, W_X)$ is the order of $\delta_{U_\eta}$, while $\mathrm{Tor}^\Lambda(Y, W_Y)$ is the order of $\delta_{U_0}$. $\square$

## 4.3. Obstruction map

Let $Y = \bigcup_{i \in I} Y_i$ be an snc scheme over $k$ (see Definition 2.3.1). We say that $Y$ has no triple intersections if $Y_i \cap Y_j \cap Y_l = \emptyset$ for all pairwise different $i, j, l \in I$.

The following map is the key player in Theorem 4.1.4, stated in the introduction.

**Definition 4.3.1.** Let $Y = \bigcup_{i \in I} Y_i$ be an snc scheme over $k$ which has no triple intersections. We fix a total order '$<$' on $I$. Let $\Lambda$ be a ring. Then we define the obstruction map

$$\Psi_Y^\Lambda \colon \bigoplus_{l \in I} \mathrm{CH}_1(Y_l, \Lambda) \longrightarrow \bigoplus_{\substack{i,j \in I \\ i<j}} \mathrm{CH}_0(Y_{ij}, \Lambda), \quad (\gamma_l)_l \mapsto \left(\gamma_i|_{Y_{ij}} - \gamma_j|_{Y_{ij}}\right)_{i,j}, \tag{4.3.1}$$

where $Y_{ij} = Y_i \cap Y_j$ for $i, j \in I$.

In the above definition, $\gamma_i|_{Y_{ij}}$ is the intersection with the divisor $Y_{ij} \subset Y_i$, see [Ful98, §2.3]. In particular, $\gamma_i|_{Y_{ij}}$ is represented by a 0-cycle supported on the set-theoretic intersection $|\gamma_j| \cap Y_{ij}$ and hence can be viewed as a class in $\mathrm{CH}_0(Y_{ij}, \Lambda)$, see also [Ful98, Convention 1.4]. We further note that we have $\mathrm{CH}_0(Y_{ij}, \Lambda) = 0$ if $Y_{ij} = \emptyset$ and we set $\gamma_i|_{Y_{ij}} = 0$ in this case.

**Theorem 4.3.2.** *Let $R$ be a discrete valuation ring with algebraically closed residue field $k$ and fraction field $K$. Let $\Lambda$ be a ring of positive characteristic $c \in \mathbb{Z}_{\geq 1}$ such that the exponential characteristic of $k$ is invertible in $\Lambda$. Let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable $R$-scheme (see Definition 2.3.1) with geometrically integral generic fibre $X = \mathcal{X} \times_R K$ and special fibre $Y = \mathcal{X} \times_R k$. Assume that $Y = \bigcup_{i \in I} Y_i$ has no triple intersections and fix a total order '$<$' on $I$. Then the cokernel of the* *map*

$$\Psi^{\Lambda}_{Y_L}\colon \bigoplus_{l\in I}\mathrm{CH}_1(Y_l\times_k L,\Lambda)\longrightarrow \bigoplus_{\substack{i,j\in I\\ i<j}}\mathrm{CH}_0(Y_{ij}\times_k L,\Lambda)$$

*from* (4.3.1) *is* $\mathrm{Tor}^{\Lambda}(\bar{X},\emptyset)$*-torsion for every field extension* $L/k$*.*

Note that the family $\mathcal{X}\to\operatorname{Spec}R$ in the above theorem is not assumed to be proper. We can always choose a relative Nagata compactification of this morphism, see [Stacks, Tag 0F41]. Replacing $\mathrm{Tor}^{\Lambda}(\bar{X},\emptyset)$ by the relative torsion order of the compactification (cf. Definition 4.2.3 and Remark 4.2.4), we can then rephrase the above theorem as follows.

**Theorem 4.3.3.** *Let $R$ be a discrete valuation ring with algebraically closed residue field $k$ and fraction field $K$. Let $\Lambda$ be a ring of positive characteristic $c\in\mathbb{Z}_{\geq 1}$ such that the exponential characteristic of $k$ is invertible in $\Lambda$. Let $\mathcal{X}\to\operatorname{Spec}R$ be a proper flat $R$-scheme with geometrically integral generic fibre $X=\mathcal{X}\times_R K$ and special fibre $Y=\mathcal{X}\times_R k$. Let $W_{\mathcal{X}}\subset\mathcal{X}$ be a closed subscheme and let $W_X\coloneqq W_{\mathcal{X}}\cap X$ and $W_Y\coloneqq W_{\mathcal{X}}\cap Y$ be the respective intersections (i.e. fibre products). Assume that the following conditions are satisfied:*

*(1) $\mathcal{X}^{\circ}\coloneqq\mathcal{X}\setminus W_{\mathcal{X}}$ is a strictly semi-stable $R$-scheme (see Definition 2.3.2);*

*(2) $Y^{\circ}\coloneqq Y\setminus W_Y=\bigcup_{i\in I}Y_i^{\circ}$ has no triple intersections.*

*Given a total order '$<$' on $I$, the cokernel of the map*

$$\Psi^{\Lambda}_{Y^{\circ}_L}\colon \bigoplus_{l\in I}\mathrm{CH}_1(Y^{\circ}_l\times_k L,\Lambda)\longrightarrow \bigoplus_{\substack{i,j\in I\\ i<j}}\mathrm{CH}_0(Y^{\circ}_{ij}\times_k L,\Lambda)$$

*from* (4.3.1) *is* $\mathrm{Tor}^{\Lambda}(\bar{X},W_{\bar{X}})$*-torsion for every field extension $L/k$, where $W_{\bar{X}}\coloneqq W_X\times_K\bar{K}$.*

**Corollary 4.3.4.** *Let the assumptions be as in Theorem 4.3.3. If $\bar{X}$ admits a $\Lambda$-decomposition of the diagonal relative to $W_{\bar{X}}$, then the map*

$$\Psi^{\Lambda}_{Y^{\circ}_L}\colon \bigoplus_{l\in I}\mathrm{CH}_1(Y^{\circ}_l\times_k L,\Lambda)\longrightarrow \bigoplus_{\substack{i,j\in I\\ i<j}}\mathrm{CH}_0(Y^{\circ}_{ij}\times_k L,\Lambda)$$

*is surjective for every field extension $L/k$.*

We illustrate the mechanism of the above corollary in the following example, where we show formally that the fact that elliptic curves admit no decomposition of the diagonal with respect to a finite set of points implies that the same holds for certain quartic surfaces. The result itself is of course a well-known consequence of [BS83] and the emphasis lies in the mechanism of the argument.

**Example 4.3.5.** Let $\Lambda$ be a ring of characteristic $c \geq 2$. Let $k$ be an algebraically closed field such that the exponential characteristic of $k$ is invertible in $\Lambda$ and let $f, g_0, g_1 \in k[x_0, x_1, x_2, x_3]$ be general homogeneous polynomials of degrees $\deg f = 4$ and $\deg g_i = 2$. Consider the degeneration $\mathcal{X} = \{t \cdot f + g_0 g_1 = 0\}$ over $R = k[[t]]$ of a smooth quartic surface into the union $Y_0 \cup Y_1$ of two smooth quadric surfaces $Y_i = \{g_i = 0\}$ which meet along a smooth elliptic curve $Z \coloneqq Y_{01} = Y_0 \cap Y_1$. Let $S \subset \mathcal{X}$ be a closed reduced subscheme which is finite and surjective over $\operatorname{Spec} R$. We further let $T_i \subset Y_i$ be the union of two lines contained in the two different rulings of the quadric surface $Y_i$ and set

$$W_{\mathcal{X}} \coloneqq S \cup T_0 \cup T_1 \cup \mathcal{X}^{\mathrm{sing}},$$

where we note that the singular locus $\mathcal{X}^{\mathrm{sing}} = \{f = g_0 = g_1 = t = 0\}$ is a finite number of points on $Z$. In the above notation, $Y_i^\circ = Y_i \setminus W_{\mathcal{X}}$ is an open subset of $Y_i \setminus T_i \simeq \mathbb{A}^2$, hence $\mathrm{CH}_1(Y_i^\circ \times_k L, \Lambda) = 0$ for $i = 0, 1$ and any field extension $L/k$. By construction, $W_{\mathcal{X}}$ meets $Z$ in finitely many points. Since the elliptic curve $Z$ does not admit a $\Lambda$-decomposition of the diagonal with respect to a finite collection of points, the group $\mathrm{CH}_0(Z^\circ \times_k L, \Lambda)$ is nontrivial for $L = k(Z)$, where $Z^\circ = Z \setminus W_{\mathcal{X}}$. It follows that $\Psi^\Lambda_{Y_L^\circ}$ is not surjective and so Corollary 4.3.4 implies that the geometric generic fibre $\bar{X}$ of our family does not admit a $\Lambda$-decomposition of the diagonal with respect to the finite set of points $S \times_R \bar{K}$.

The following example extends the argument in the previous example to the non-trivial case of quartic fivefolds, which was the main result in [PS23]. We leave some details to the reader; similar constructions and arguments will be given with full details in the proof of Theorem 4.1.1 below.

**Example 4.3.6.** Let $k$ be an algebraically closed field of characteristic different from 2 and let $\Lambda = \mathbb{Z}/2$. Consider the homogeneous polynomial

$$f = x_0^2 y_1^2 + x_1 x_2 y_2^2 + x_0 x_2 y_3^2 + x_0 x_1 g + x_0^3 z + x_0^2 y_1 w \in k[x_0, x_1, x_2, y_1, y_2, y_3, z, w],$$

where $g \coloneqq x_0^2 + x_1^2 + x_2^2 - 2x_0x_1 - 2x_0x_2 - 2x_1x_2$. We consider the degeneration

$$\mathcal{X} \coloneqq \{tx_0^2 + zw = f = 0\} \subset \mathbb{P}^7_R \to \operatorname{Spec} R$$

over $R = k[[t]]$ of a singular $(2,4)$-complete intersection $X$ in $\mathbb{P}^7_{k((t))}$ into a union $Y = Y_0 \cup Y_1$ of two singular quartic fivefolds intersecting in the singular integral quartic fourfold

$$Z \coloneqq Y_{01} = \{x_0^2y_1^2 + x_1x_2y_2^2 + x_0x_2y_3^2 + x_0x_1g = 0\} \subset \mathbb{P}^5_k,$$

which is birational to the quadric surface bundle example in [HPT18a, Example 8]. Let $S \subset \mathcal{X}$ be the closure of a finite set of points in $X$ and let

$$W_{\mathcal{X}} \coloneqq S \cup \{x_0x_1x_2y_1 = 0\} \cup \operatorname{Sing} Z \subset \mathcal{X}.$$

One can check that $\mathcal{X}^\circ \coloneqq \mathcal{X} \setminus W_{\mathcal{X}}$ is strictly semi-stable. The singular quartic fivefold $Y_i$ $(i = 0,1)$ is birational to $\mathbb{P}^5$ via the projection from the singular point of multiplicity 3 given by $x_i = 0$ and $y_j = 0$ for all $i,j$ and $z = 1$ (resp. $w = 1$). The restriction of this projection to the open subset $Y_i^\circ = Y_i \setminus W_{\mathcal{X}}$ gives an isomorphism with an open subset of $\mathbb{A}^5 = \mathbb{P}^5 \setminus \{x_0 = 0\}$. Hence, $\mathrm{CH}_1(Y_i^\circ \times_k L) = 0$ for every field extension $L/k$. Recall the unramified class $\alpha = (x_1/x_0, x_2/x_0) \in H^2_{nr}(k(Z), \mathbb{Z}/2)$ from [HPT18a, Proposition 11]. A direct computation (using e.g. [Sch19b, Theorem 2.3]) shows that this class vanishes on the generic points of the divisors $\{x_0x_1x_2 = 0\}$ and $\{y_1 = 0\}$ (the latter uses that $\langle \frac{g}{x_0^2} \rangle$ is a subform of the quadratic form $\langle 1, \frac{x_1}{x_0} \rangle$ over $k(\mathbb{P}^2)$). We then pass to a resolution or alteration of $Z$ and apply the vanishing result in [Sch19b, Proposition 5.1] to the unramified class from [HPT18a, Proposition 11], to conclude via an "action of correspondences" argument that $Z$ does not admit a $\Lambda$-decomposition of the diagonal relative to $W_Z = W_{\mathcal{X}} \cap Z = (S \cap Z) \cup \{x_0x_1x_2y_1 = 0\} \cup \operatorname{Sing} Z \subset Z$; see Theorem 4.5.1 below for more details on this type of argument. Thus, $\mathrm{CH}_0(Z^\circ \times k(Z), \Lambda)$ is nontrivial, where $Z^\circ = Z \setminus W_Z$. It follows that the map $\Psi^\Lambda_{Y^\circ_{k(Z)}}$ in Corollary 4.3.4 is not surjective. Consequently, $\bar{X}$ does not admit a $\Lambda$-decomposition of the diagonal relative to $W_{\bar{X}} = (W_{\mathcal{X}} \cap X)_{\bar{K}}$. The substitution $y_1 = zy_1/x_0$ and $w = -tx_0^2/z$ shows that the $(2,4)$ complete intersection $X$ is birational to the singular quartic hypersurface

$$X' = \{z^2y_1^2 + x_1x_2y_2^2 + x_0x_2y_3^2 + x_0x_1g + x_0^3z - tx_0^3y_1 = 0\} \subset \mathbb{P}^6_{k((t))}.$$

One checks that the restriction of the induced birational map $X \dashrightarrow X'$ to $X \setminus W_{\mathcal{X}}$ yields an isomorphism onto its image. It follows that, even after base change to an algebraic closure of $k((t))$, $X'$ does not admit a decomposition of the diagonal relative to the union of a finite set of points with $\{x_0x_1x_2y_1z = 0\} \subset X'$. This produces a singular quartic fivefold that does not admit a $\Lambda$-decomposition of the diagonal relative to a finite set of points (union its singular locus). We can then apply Lemma 4.2.9 to conclude that any smooth quartic fivefold that degenerates to $X'$ does not admit a decomposition of the diagonal relative to a finite set of points. Altogether, this yields a simplified argument for the main result in [PS23].

**Remark 4.3.7.** In order to prove Theorem 4.1.1, we would like to replace in the above argument the example [HPT18a, Example 8] with the higher dimensional examples in [Sch19b, Section 4 and 7]. This does not work directly as the unramified class $\alpha$ from [Sch19b, Proposition 5.1] does not vanish along the divisor $\{y_1 = 0\}$. We circumvent this problem by introducing an additional parameter $\lambda$ in the family, see Section 4.4. Further technical difficulties appear because we aim to apply the above argument inductively, where we degenerate to $Y_0 \cup Y_1$ such that $Y_0 \cap Y_1$ is birational to the example of lower dimension we had treated before.

The following simplified version of the Theorem 4.3.2 is used in Chapter 5.

**Theorem 4.3.8.** *Let $R$ be a discrete valuation ring with algebraically closed residue field $k$ and fraction field $K$. Let $\Lambda$ be a ring of positive characteristic $c \in \mathbb{Z}_{\geq 1}$, i.e. every $\lambda \in \Lambda$ is $c$-torsion, and assume that $c \in k^*$. Let $\mathcal{X}$ be a flat separated $R$-scheme of finite type with geometrically integral generic fibre $X = \mathcal{X}_K$ and special fibre $Y = \mathcal{X}_k$. Let $W_{\mathcal{X}} \subset \mathcal{X}$ be a closed subscheme and we denote by $W_V := W_{\mathcal{X}} \cap V$ the scheme-theoretic intersection with a subscheme $V \subset \mathcal{X}$. Assume the following:*

*(1) $Y$ consists of two components $Y_0, Y_1$ ($\dim Y_i \geq 2$) such that $Z := Y_0 \cap Y_1$ is integral;*

*(2) $\mathcal{X}^\circ := \mathcal{X} \setminus W_{\mathcal{X}}$ is a strictly semi-stable $R$-scheme, see Definition 2.3.2;*

*(3) the variety $Y_i \setminus W_{Y_i}$ is isomorphic to an open subscheme of $\mathbb{A}_k^{\dim Y_i}$ for $i = 0, 1$.*

*Then we have*
$$\mathrm{Tor}^\Lambda(Z, W_Z) \mid \mathrm{Tor}^\Lambda(X \times_K \bar{K}, W_X \times_K \bar{K}).$$

*Proof.* We apply Theorem 4.3.2 to the family $\mathcal{X}^\circ \to \operatorname{Spec} R$. Then the assumption (3) implies that the group
$$\mathrm{CH}_0(Z \setminus W_Z \times_k L, \Lambda)$$

is $\mathrm{Tor}^{\Lambda}(\bar{X} \setminus W_{\bar{X}}, \emptyset)$-torsion for every field extension $L/k$. In particular,

$$\mathrm{Tor}^{\Lambda}(Z, W_Z) \mid \mathrm{Tor}^{\Lambda}(\bar{X}, W_{\bar{X}}),$$

which finishes the proof of the theorem. □

### 4.3.1. Comparison to the map in [PS23]

We recall the obstruction map from [PS23] in the generality needed in this chapter. Let $\Lambda$ be a ring and let $Y = \bigcup_{i \in I} Y_i$ be an snc scheme over a field $k$. Then we consider for $i, j \in I$ the homomorphism

$$\Phi^{\Lambda}_{Y_i, Y_j} \colon \mathrm{CH}_1(Y_i, \Lambda) \longrightarrow \mathrm{CH}_0(Y_j, \Lambda), \quad \gamma_i \mapsto \begin{cases} \iota_{ij,j\,*}(\gamma_i|_{Y_{ij}}) & \text{if } i \neq j, \\ -\sum\limits_{l \neq i} \iota_{il,i\,*}(\gamma_i|_{Y_{il}}) & \text{if } i = j, \end{cases} \tag{4.3.2}$$

where $\iota_{ij,i} \colon Y_{ij} := Y_i \cap Y_j \to Y_i$ is the natural inclusion and $\gamma_i|_{Y_{ij}} := \iota^*_{ij,i}\gamma$ is the pullback along the regular embedding $\iota_{ij,i}$ of the Cartier divisor $Y_{ij}$ in $Y_i$, see [PS23, Definition 3.1 and Lemma 3.2]. We additionally set

$$\Phi^{\Lambda}_{Y, Y_j} := \sum_{i \in I} \Phi^{\Lambda}_{Y_i, Y_j} \colon \bigoplus_{i \in I} \mathrm{CH}_1(Y_i, \Lambda) \longrightarrow \mathrm{CH}_0(Y_j, \Lambda)$$

for $j \in I$ and

$$\Phi^{\Lambda}_{Y} := \sum_{j \in I} \Phi^{\Lambda}_{Y, Y_j} \colon \bigoplus_{i \in I} \mathrm{CH}_1(Y_i, \Lambda) \longrightarrow \bigoplus_{j \in I} \mathrm{CH}_0(Y_j, \Lambda).$$

**Lemma 4.3.9.** *Let $R$ be a discrete valuation ring with residue field $k$ and let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable $R$-scheme with special fibre $Y := \mathcal{X} \times_R k$. Then for $i, j \in I$ and $\gamma_i \in \mathrm{CH}_1(Y_i, \Lambda)$, we have*

$$\Phi^{\Lambda}_{Y_i, Y_j}(\gamma_i) = \iota_j^* \iota_{i\,*}\gamma_i \in \mathrm{CH}_0(Y_j, \Lambda),$$

*where $\iota_i \colon Y_i \to \mathcal{X}$ is the natural inclusion.*

*Proof.* This follows from [Ful98, Theorem 6.2], see also [PS23, Lemma 3.2]. □

The effect of base changes for $\Phi$ has previously been studied in [PS23, Section 4] and Theorem 3.1.1; some version of $\Psi$ appeared implicitly in the proof of [PS23, Proposition 4.4].

Proposition 4.3.12 below compares $\Phi$ and $\Psi$ and shows that the morphism $\Psi$ introduced in (4.3.1) has excellent properties under base changes. This will play an important role in the proof of Theorem 4.3.2 below.

Recall that the dual graph $G = (V, E)$ of an snc scheme $Y$ without triple intersections is the graph whose vertices $V$ correspond to the irreducible components of $Y$ and whose edges $E$ encode the codimension 1 subvarieties of $Y$. Moreover, we explicitly fix a total order '$<$' on the set of vertices $V$. We denote for a vertex $v \in V$ and an edge $e \in E$ the corresponding irreducible subvariety by $Y_v$ and $Y_e$, respectively. For an edge $e \in E$, we denote its end points by $v(e), w(e) \in V$ and assume that $v(e) < w(e)$. (Note that every edge in $G$ has two distinct vertices.)

**Lemma 4.3.10.** *Let $\Lambda$ be a ring of finite characteristic $c \in \mathbb{Z}_{\geq 1}$. Let $R$ be a discrete valuation ring with fraction field $K$ and algebraically closed residue field $k$. Let $\mathcal{X} \to \operatorname{Spec} R$ be a strictly semi-stable $R$-scheme whose special fibre $Y := \mathcal{X} \times_R k$ has no triple intersections with dual graph $G = (V, E)$. Let $R'/R$ be a finite ramified extension of discrete valuation rings such that the ramification index $r$ is divisible by $c$. Let $\mathcal{X}' \to \mathcal{X} \times_R R'$ be a resolution as in Proposition 2.3.3. Then the following holds:*

*(1) $\mathcal{X}'$ is strictly semi-stable and its special fibre $Y'$ has no triple intersections. The dual graph $G' = (V', E')$ of $Y'$ consists of the vertices $V' = V \cup (E \times \{1, \dots, r-1\})$ and edges $(v', w') \in V' \times V'$ of the form*

$$\begin{aligned} v' &= (e, n), & w' &= (e, n+1) & &\textit{for } e \in E,\ n \in \{1, \dots, r-2\}, \\ v' &= v(e), & w' &= (e, 1) & &\textit{for } e \in E,\ \textit{or} \\ v' &= (e, r-1), & w' &= w(e) & &\textit{for } e \in E. \end{aligned}$$

*(2) The components $Y'_v$ for $v \in V$ are isomorphic to $Y_v$ and the components $Y'_{(e,n)}$ for $e \in E$ and $1 \leq n < r$ are $\mathbb{P}^1$-bundles over $Y_e$ with two disjoint sections*

$$\begin{aligned} s_{(e,n)} &\colon Y'_{(e,n)} \cap Y'_{(e,n+1)} \longrightarrow Y'_{(e,n)}, \\ s'_{(e,n)} &\colon Y'_{(e,n-1)} \cap Y'_{(e,n)} \longrightarrow Y'_{(e,n)}, \end{aligned}$$

*given by the natural inclusion, where we set $Y'_{(e,0)} = Y'_{v(e)}$ and $Y'_{(e,r)} = Y'_{w(e)}$.*

*Proof.* The geometry of the resolution after finite base-change is explained for example in [PS23, Section 4.2] for chains. The same arguments work for any special fibre that has no triple intersections, see also Proposition 3.3.12. $\square$

**Remark 4.3.11.** The graph $(V', E')$ is obtained by subdividing each edge of $(V, E)$ into $r$ pieces, see for example Example 3.6.1.

**Proposition 4.3.12.** *In the notation of Lemma 4.3.10, let $L/k$ be a field extension. If the cokernel of*

$$\Phi^{\Lambda}_{Y'_L} : \bigoplus_{v' \in V'} \mathrm{CH}_1(Y'_{v'} \times_k L, \Lambda) \longrightarrow \bigoplus_{w' \in V'} \mathrm{CH}_0(Y'_{w'} \times_k L, \Lambda)$$

*is $m$-torsion for some integer $m$, then the cokernel of the map*

$$\Psi^{\Lambda}_{Y_L} : \bigoplus_{l \in V} \mathrm{CH}_1(Y_l \times_k L, \Lambda) \longrightarrow \bigoplus_{\substack{i,j \in V \\ i<j}} \mathrm{CH}_0(Y_{ij} \times_k L, \Lambda)$$

*from (4.3.1) is $m$-torsion.*

*Proof.* For ease of notation, we will deal with the case $L = k$ in what follows; the general case follows verbatim via the same argument.

Recall that $q_{(e,n)} : Y'_{(e,n)} \to Y_e$ is a $\mathbb{P}^1$-bundle for $1 \leq n < r$ by item (2) in Lemma 4.3.10. Thus there exists isomorphisms

$$\mathrm{CH}_1(Y'_{(e,n)}, \Lambda) \simeq \mathrm{CH}_1(Y_e, \Lambda) \oplus q^*_{(e,n)} \mathrm{CH}_0(Y_e, \Lambda), \tag{4.3.3}$$

$$\mathrm{CH}_0(Y'_{(e,n)}, \Lambda) \simeq \mathrm{CH}_0(Y_e, \Lambda), \tag{4.3.4}$$

see [Ful98, Theorem 3.3 (b)]. Note that the subspace $q^*_{(e,n)} \mathrm{CH}_0(Y_e, \Lambda)$ is canonical, while the subspace $\mathrm{CH}_1(Y_e, \Lambda) \subset \mathrm{CH}_1(Y'_{(e,n)}, \Lambda)$ depends on a choice of a section of $q_{(e,n)} : Y'_{(e,n)} \to Y_e$.

**Step 1.** We will show that for every $\gamma' \in \bigoplus_{v' \in V'} \mathrm{CH}_1(Y'_{v'}, \Lambda)$, there exists another class $\gamma \in \bigoplus_{v' \in V'} \mathrm{CH}_1(Y'_{v'}, \Lambda)$ with the same image $\Phi^{\Lambda}_{Y'}(\gamma) = \Phi^{\Lambda}_{Y'}(\gamma')$, such that the component $\gamma_{(e,n)} \in \mathrm{CH}_1(Y'_{(e,n)}, \Lambda)$ of $\gamma$ satisfies

$$\gamma_{(e,n)} \in q^*_{(e,n)} \mathrm{CH}_0(Y_e, \Lambda) \subset \mathrm{CH}_1(Y'_{(e,n)}, \Lambda) \tag{4.3.5}$$

for each $(e, n) \in E \times \{1, \ldots, r-1\}$.

This follows from the argument in [PS23, Lemma 4.3], which we explain for the convenience of the reader. Let $\gamma' = (\gamma'_{v'})_{v'} \in \bigoplus_{v' \in V'} \mathrm{CH}_1(Y'_{v'}, \Lambda)$ be a collection of one-cycles and let $(e, n) \in E \times \{1, \ldots, r-1\}$. Using (4.3.3), we can write $\gamma'_{(e,n)}$ as

$$\gamma'_{(e,n)} = q^*_{(e,n)} \alpha_{(e,n)} + s_{(e,n)*} \zeta_{(e,n)},$$

for some $\alpha_{(e,n)} \in \mathrm{CH}_0(Y_e, \Lambda)$ and some $\zeta_{(e,n)} \in \mathrm{CH}_1(Y_e, \Lambda)$, where $s_{(e,n)}$ is the section given in item (2) of Lemma 4.3.10. Lemma 4.3.9 implies that

$$\Phi^{\Lambda}_{Y'}(s_{(e,n)*}\zeta_{(e,n)}) = \Phi^{\Lambda}_{Y'}(s'_{(e,n+1)*}\zeta_{(e,n)}),$$

where $s'_{(e,n+1)} \colon Y'_{(e,n)} \cap Y'_{(e,n+1)} \to Y'_{(e,n+1)}$ is the natural inclusion. Thus

$$\Phi^{\Lambda}_{Y'}(\gamma') = \Phi^{\Lambda}_{Y'}(\gamma''),$$

where $\gamma'' = (\gamma''_{v'})_{v'} \in \bigoplus_{v' \in V'} \mathrm{CH}_1(Y'_{v'}, \Lambda)$ is given by

$$\gamma''_{v'} = \begin{cases} q^*_{(e,n)}\alpha_{(e,n)} & \text{if } v' = (e,n), \\ \gamma'_{(e,n+1)} + s'_{(e,n+1)*}\zeta_{(e,n)} & \text{if } v' = (e,n+1), \\ \gamma'_v & \text{otherwise,} \end{cases}$$

where we set $\gamma'_{(e,r)} = \gamma'_{w(e)} \in \mathrm{CH}_1(Y'_{w(e)}, \Lambda)$. Applying this argument for every edge $e \in E$ and $1 \le n \le r-1$ (in increasing order) finishes Step 1.

Since $Y$ is an snc scheme without triple intersections, we have $Y_e \cap Y_{e'} = \emptyset$ for all different $e, e' \in E$. In particular,

$$\bigoplus_{\substack{i,j \in V \\ i<j}} \mathrm{CH}_0(Y_{ij}, \Lambda) = \bigoplus_{e \in E} \mathrm{CH}_0(Y_e, \Lambda).$$

Using this identification, there is a natural projection homomorphism

$$\mathrm{pr}_e \colon \bigoplus_{\substack{i,j \in V \\ i<j}} \mathrm{CH}_0(Y_{ij}, \Lambda) \longrightarrow \mathrm{CH}_0(Y_e, \Lambda)$$

for every $e \in E$. We denote the composition $\mathrm{pr}_e \circ \Psi^{\Lambda}_Y$ by $\Psi^{\Lambda}_{Y,Y_e}$.

**Step 2.** Let $\gamma = (\gamma_{v'})_{v'} \in \bigoplus_{v' \in V'} \mathrm{CH}_1(Y'_{v'}, \Lambda)$ be a one-cycle satisfying (4.3.5) and let $q \colon Y' \to Y$ be the natural morphism. We will show that for every $e \in E$

$$\sum_{n=1}^{r-1} n \cdot \Phi^{\Lambda}_{Y', Y'_{(e,n)}}(\gamma) = \Psi^{\Lambda}_{Y,Y_e}(q_*\gamma) \in \mathrm{CH}_0(Y_e, \Lambda), \tag{4.3.6}$$

where we view the zero-cycles on the left hand side as zero-cycles on $Y_e$ using (4.3.4). To prove the above claim, note that by assumption there exists $\alpha_{(e,n)} \in \mathrm{CH}_0(Y_e, \Lambda)$

for every $(e, n) \in E \times \{1, \dots, r-1\}$ such that

$$\gamma_{(e,n)} = q^*_{(e,n)} \alpha_{(e,n)} \in \mathrm{CH}_1(Y'_{(e,n)}, \Lambda).$$

To simplify the formulas below, we set additionally

$$\begin{aligned} \alpha_{(e,0)} &:= \gamma_{v(e)}|_{Y_e} \in \mathrm{CH}_0(Y_e, \Lambda), \\ \alpha_{(e,r)} &:= \gamma_{w(e)}|_{Y_e} \in \mathrm{CH}_0(Y_e, \Lambda), \end{aligned}$$

where $\gamma_{v(e)}|_{Y_e}$ is the pullback of $\gamma_{v(e)}$ along the regular embedding $Y_e \hookrightarrow Y_{v(e)}$ of the Cartier divisor $Y_e \subset Y_{v(e)}$. Note that we used here the isomorphisms $Y'_{v(e)} \simeq Y_{v(e)}$. By (4.3.2), we see that for such a collection of one-cycles $\gamma = (\gamma_{v'})_{v'}$

$$\Phi^{\Lambda}_{Y', Y'_{(e,n)}}(\gamma) = -2\alpha_{(e,n)} + \alpha_{(e,n-1)} + \alpha_{(e,n+1)} \in \mathrm{CH}_0(Y_e, \Lambda) \simeq \mathrm{CH}_0(Y'_{(e,n)}, \Lambda)$$

for $e \in E$ and $1 \leq n < r$. The following computation then shows the claim in Step 2

$$\begin{aligned} \sum_{n=1}^{r-1} n \cdot \Phi^{\Lambda}_{Y', Y'_{(e,n)}}(\gamma) &= \sum_{n=1}^{r-1} n \left( -2\alpha_{(e,n)} + \alpha_{(e,n-1)} + \alpha_{(e,n+1)} \right) \\ &= \alpha_{(e,0)} - r\alpha_{(e,r-1)} + (r-1)\alpha_{(e,r)} + \sum_{n=1}^{r-2} (-2n + n + 1 + n - 1)\alpha_{(e,n)} \\ &= \alpha_{(e,0)} - \alpha_{(e,r)} \\ &= \gamma_{v(e)}|_{Y_e} - \gamma_{w(e)}|_{Y_e} \\ &= \Psi^{\Lambda}_{Y, Y_e}(q_* \gamma), \end{aligned}$$

where we used in the third equality that $r$ is divisible by the characteristic $c$ of $\Lambda$. We finish the proof of the proposition. To this end, let

$$z = (z_e)_e \in \bigoplus_{e \in E} \mathrm{CH}_0(Y_e, \Lambda)$$

be a collection of zero-cycles. By assumption, there exists a one-cycle $\gamma \in \bigoplus_{v' \in V'} \mathrm{CH}_1(Y'_{v'}, \Lambda)$ such that $\Phi^{\Lambda}_{Y'}(\gamma) = \beta$, where $\beta$ is the collection $(\beta_{v'})_{v'} \in \bigoplus_{v' \in V'} \mathrm{CH}_0(Y'_{v'}, \Lambda)$ with

$$\beta_{v'} = \begin{cases} m \cdot z_e & \text{for } v' = (e, 1), \\ 0 & \text{otherwise.} \end{cases}$$

By Step 1, we can assume that $\gamma$ satisfies the condition (4.3.5). Then Step 2 shows that $m \cdot z = \Psi_Y^\Lambda(q_*\gamma)$ is contained in the image of $\Psi_Y^\Lambda$, as we want. □

**Remark 4.3.13.** The key point in the above proof is (4.3.6). In the chain of equalities showing (4.3.6), we used that $\Lambda$ has positive characteristic $c$ and $r$ is divisible by $c$. We do not know how to perform this step integrally, despite the fact that the (more general) analysis of $\Phi$ under base changes carried out in Section 3.4 does in fact work integrally.

**Remark 4.3.14.** In [PS23], the obstruction morphism $\Phi_Y^{\mathbb{Z}}$ is studied for strictly semi-stable degenerations $\mathcal{X} \to \operatorname{Spec} R$ that are proper. Under this assumption, the image of $\Phi_Y^{\mathbb{Z}}$ is contained in the kernel of the degree map. The obstruction to the existence of a decomposition of the diagonal used in [PS23] is the cokernel of $\Phi_Y^{\mathbb{Z}}$, viewed as a map to the subspace of degree-zero classes. In the above discussion, properness is dropped and so we cannot talk about the degree anymore. This is the reason why we work directly with the cokernel of $\Phi$ and $\Psi$, respectively. An important difference between $\Phi$ and $\Psi$ is the fact that $\Phi$ maps to the Chow groups of zero-cycles of the components, while $\Psi$ maps to $\mathrm{CH}_0$ of the intersections $Y_{ij}$ of two irreducible components of $Y$.

### 4.3.2. Proof of Theorem 4.3.2

*Proof of Theorem 4.3.2.* Let $m := \operatorname{Tor}^\Lambda(\bar{X}, \emptyset)$. Then there exists a finite field extension $F/K$ such that $\operatorname{Tor}^\Lambda(X_F, \emptyset) = m$.

A suitable localization $R'$ of the integral closure of $R$ in $F$ is also a discrete valuation ring with fraction field $F$ and residue field $k$. (Note that $k$ is algebraically closed.) Up to replacing $R'$ by a ramified extension, we can assume that the ramification index $r$ of $R'/R$ is divisible by the characteristic $c$ of $\Lambda$.

Recall that the special fibre of $\mathcal{X} \to \operatorname{Spec} R$ is an snc scheme $Y$ which has no triple intersections by assumption and we denote its dual graph by $(V, E)$. By Lemma 4.3.10, there exists a resolution $\tilde{\mathcal{X}} \to \mathcal{X} \times_R R'$ by repeatedly blowing-up the non-Cartier components of the special fibre. The generic fibre of $\tilde{\mathcal{X}} \to \operatorname{Spec} R'$ is isomorphic to $X_F$ and the special fibre $Y'$ has no triple intersections and its dual graph $(V', E')$ is as in Lemma 4.3.10 (1). It then suffices by Proposition 4.3.12 to show that $\operatorname{coker} \Phi_{Y'_L}^\Lambda$ is $m$-torsion for all field extensions $L/k$.

Let $L/k$ be a field extension. By inflation of local rings (see e.g. [Bou06, Chapter IX, Appendice §2, Corollaire du Théorème 1 and Exercice 4]), there exists an unramified extension of discrete valuation rings $A/R'$ such that the induced extension of residue fields is $L/k$. Passing to the completion, we may in addition assume that $A$ is complete. We consider the base-change

$$\tilde{\mathcal{X}}_A \coloneqq \tilde{\mathcal{X}} \times_{R'} A \to \operatorname{Spec} A,$$

which is a strictly semi-stable $A$-scheme, see e.g. [Har01, Proposition 1.3]. We aim to show that the cokernel of the map

$$\Phi^{\Lambda}_{Y'_L} \colon \bigoplus_{v' \in V'} \mathrm{CH}_1(Y'_{v',L}, \Lambda) \longrightarrow \bigoplus_{w' \in V'} \mathrm{CH}_0(Y'_{w',L}, \Lambda)$$

defined in (4.3.2) is $m$-torsion, where $Y'_{v',L} \coloneqq Y'_{v'} \times_k L$. Let

$$z = (z_{w'})_{w'} \in \bigoplus_{w' \in V'} \mathrm{CH}_0(Y'_{w',L}, \Lambda)$$

be a collection of zero-cycles. (By a moving lemma, we can assume that no $z_{w'}$ lies in the intersection of $Y'_{w'}$ with another component $Y'_{v'}$.) By Hensel's lemma, see [EGAIV, Théorème 18.5.17], there exists a horizontal one-cycle $h \in Z_1(\tilde{\mathcal{X}}_A, \Lambda)$ such that

$$h|_{Y'_{w',L}} = z_{w'} \in \mathrm{CH}_0(Y'_{w',L}, \Lambda) \tag{4.3.7}$$

holds for all $w'$ already on the level of cycles (without rational equivalence). Recall that $m$ is the $\Lambda$-torsion order of $X_F$ with respect to the empty set. Lemma 4.2.8 together with Nagata's compactification theorem thus implies that $\mathrm{CH}_0(X_F \times_F F', \Lambda)$ is $m$-torsion for all field extension $F'/F$. It follows that the restriction of $m \cdot h$ to the generic fibre $\tilde{\mathcal{X}}_A \times_A F' = X_F \times_F F'$ vanishes, where $F'$ is the fraction field of $A$. Thus the horizontal one-cycle $m \cdot h$ is rationally equivalent to a cycle $\gamma$ supported on the special fibre $Y'_L$ by the localization exact sequence, see [Ful98, §1.8]. Hence, we see from (4.3.7) and Lemma 4.3.9 that $m \cdot z = \Phi^{\Lambda}_{Y'_L}(\gamma)$ is contained in the image of $\Phi^{\Lambda}_{Y'_L}$. This shows that the cokernel of the map $\Phi^{\Lambda}_{Y'_L}$ is $m$-torsion, which finishes the proof of the theorem. $\square$

**Remark 4.3.15.** The assumption that the special fibre has no triple intersections in Theorem 4.3.2 appears to be crucial. In particular, there seems to be no useful generalization of $\Psi$ to snc schemes with deeper strata, despite the fact that the definition of $\Phi$ in (4.3.2) makes sense in more generality, see [PS23, Section 3]. Indeed, if $Z = Y_{i_1} \cap \cdots \cap Y_{i_n}$ is a strata of the special fibre $Y$, then we can always perform an $n : 1$ base change followed by a resolution as in Proposition 2.3.3 to arrive at a special fibre $\tilde{Y}$ that contains a component $P_Z$ that is birational to a projective bundle over $Z$. However, for $n \geq 3$, one can show via similar arguments as in [Lan24] that the diagonal point of $P_Z$ will automatically be in the image of $\Phi_{\tilde{Y}_{k(P_Z)}}$. (The key difference to $n \leq 2$ is that the blow-up of a component $Y_{i_j}$ along $Z$ contains a positive dimensional projective bundle over $Z$ if $n \geq 3$.) Hence, the strategy for disproving (retract) rationality in [PS23, Lan24] cannot be applied to strata given by the intersection of more than 2 components.

## 4.4. Double cone construction

In this section we consider an explicit degeneration of a variety birational to a degree $d$ hypersurface $X'$ into a union of two rational varieties whose intersection $Z$ is a degree $d$ hypersurface of lower dimension. We aim to apply Theorem 4.3.3 to this particular family and show that the $\Lambda$-torsion order of $Z$ divides the $\Lambda$-torsion order of $X'$. In particular, we can inductively increase the dimension of retract irrational hypersurfaces.

Let $k = \overline{k_0(\lambda)}$ be an algebraic closure of the purely transcendental field extension $k_0(\lambda)$ of an algebraically closed field $k_0$. Write $N = n + r + s$ for some integers $n, r, s \in \mathbb{Z}_{\geq 0}$ and consider integers $d \geq 4$ and $l \geq 1$ such that $2l \leq d$. We denote the homogeneous coordinates of $\mathbb{P}^{N+3}_k$ by $x_0, \dots, x_n, y_1, \dots, y_{r+1}, z_1, \dots, z_s, z, w$. Let

$$f, a_0, a_1, \dots, a_l \in k_0[x_0, \dots, x_n, y_1, \dots, \hat{y_j}, \dots, y_{r+1}, z_1, \dots, z_s] \tag{4.4.1}$$

be homogeneous polynomials of degree $\deg f = d$ and $\deg a_i = d - 2i$ which do not contain the variable $y_j$ for some $1 \leq j \leq r + 1$. Assume that

$$f + a_0 \in k[x_0, \dots, x_n, y_1, \dots, \hat{y_j}, \dots, y_{r+1}, z_1, \dots, z_s] \text{ is irreducible.} \tag{4.4.2}$$

Let $R := k[t]_{(t)}$ be the local ring of $\mathbb{A}^1$ at the origin and consider the following complete intersection $R$-scheme in $\mathbb{P}^{N+3}_R$

$$\mathcal{X} := \left\{ f + \sum_{i=0}^{l} a_i x_0^i y_j^i + x_0^{d-1} z + x_0^{d-2}(\lambda y_j + x_0) w = t x_0^2 + z w = 0 \right\}. \tag{4.4.3}$$

**Lemma 4.4.1.** *The singular locus of $\mathcal{X}$ in (4.4.3) is contained in $\{x_0 = 0\} \subset \mathcal{X}$.*

*Proof.* Consider the part of the Jacobian given by the derivatives $\partial_t$ and $\partial_z$

$$\begin{pmatrix} 0 & x_0^{d-1} \\ x_0^2 & w \end{pmatrix}.$$

As the singular locus is given by the vanishing of all $2 \times 2$ minors of the Jacobian, we see that it is contained in $\{x_0 = 0\}$ as claimed. ☐

The generic fibre of the family $\mathcal{X} \to \operatorname{Spec} R$ in (4.4.3) is birational to a degree $d$ hypersurface. For the inductive argument to work, it will be important to understand the corresponding birational map, which is the content of the following lemma, where we denote by $K \coloneqq k(t)$ the fraction field of $R = k[t]_{(t)}$.

**Lemma 4.4.2.** *The generic fibre $X = \mathcal{X} \times_R K$ of the family (4.4.3) is birational to a geometrically integral degree $d$ hypersurface $X'$ of the form*

$$X' = \left\{ f + \sum_{i=0}^{l} a_i' y_j^i = 0 \right\} \subset \mathbb{P}_K^{N+2}, \tag{4.4.4}$$

*where $f$ is as in (4.4.1) and $a_0', \dots, a_l' \in K[x_0, \dots, x_n, y_1, \dots, \hat{y}_j, \dots, y_{r+1}, z_1, \dots, z_{s+1}]$ are homogeneous polynomials of degree $\deg a_i' = d - i$. (An explicit formula for them is given in (4.4.6) below.) Moreover, they satisfy the following properties:*

1. *The birational map induces an isomorphism between the open subsets $\{x_0 \neq 0\} \subset X$ and $\{x_0 z_{s+1} \neq 0\} \subset X'$;*
2. *If there exists $e \in \mathbb{N}$ such that $x_0^{ie} \mid a_i$ for all $i = 0, \dots, l$, then $x_0^{ie} \mid a_i'$ for all $i = 0, \dots, l$;*
3. *If all $a_0, \dots, a_l$ do not contain one of the coordinates $x_1, \dots, x_n$, $y_1, \dots, y_{r+1}$, or $z_1, \dots, z_s$, then so do all $a_0', \dots, a_l'$;*
4. *The polynomial $f + a_0'$ is irreducible over $\bar{K}$.*

*Proof.* The parameter $t$ is nonzero on the generic fibre $X$ of (4.4.3). We then work on the open subset $\{x_0 z \neq 0\}$ and perform the substitution $w = -t x_0^2 z^{-1}$ to obtain the equation

$$f + \sum_{i=0}^{l} a_i x_0^i y_j^i + x_0^{d-1} z - x_0^{d-2} (\lambda y_j + x_0) t x_0^2 z^{-1} = 0. \tag{4.4.5}$$

After the change of coordinates $y_j \mapsto y_j z/x_0 - \lambda^{-1}x_0$ we arrive at the degree $d$ hypersurface given by the vanishing of the polynomial

$$f + \sum_{i=0}^{l} a_i(y_j z - \lambda^{-1}x_0^2)^i + x_0^{d-1}z - t\lambda x_0^{d-1}y_j = 0.$$

Reordering the terms and renaming the coordinate $z$ to $z_{s+1}$, yields the claim that $X$ is birational to a degree $d$ hypersurface $X'$ of the form (4.4.4), where

$$a_i' = z_{s+1}^i\left(\sum_{m=i}^{l}\binom{m}{i}(-\lambda^{-1})^{m-i}x_0^{2m-2i}a_m\right) - \delta_{i,1}t\lambda x_0^{d-1} + \delta_{i,0}x_0^{d-1}z_{s+1}. \quad (4.4.6)$$

This shows (2) and (3). Item (1) follows immediately from the above construction. Indeed, the coordinate transformation $y_j \mapsto y_j z/x_0 - \lambda^{-1}x_0$ is invertible on the set $\{x_0 z \neq 0\}$ with inverse given by $y_j \mapsto x_0(y_j + \lambda^{-1}x_0)/z$. Thus the birational map induces an isomorphism between the open subsets $\{x_0 \neq 0\} = \{x_0 z \neq 0\} \subset X$ and $\{x_0 z_{s+1} \neq 0\} \subset X'$. (Note that we renamed the $z$-coordinate to $z_{s+1}$.) Next we prove (4). Since $a_0'$ contains the variable $z_{s+1}$ linearly by (4.4.6) and $f$ does not contain $z_{s+1}$, the condition (4.4.2) implies that $f + a_0'$ is irreducible, as claimed. The hypersurface $X'$ is geometrically integral by (4). □

**Remark 4.4.3.** The name "double cone construction" is taken from [Moe23]; it is reflected by the fact that the generic fibre of our degeneration can birationally be described by the equation (4.4.5) above, which contains the variable $z$ and its inverse linearly and so its Newton polytope is a double cone. We note however that the degenerations that we use in this chapter in general do not have toric singularities (e.g. because the singular hypersurfaces in [Sch19b] do not have toric singularities) and hence is different from the degenerations suitable for the method of [NO22, Moe23].

**Corollary 4.4.4.** *The generic fibre $X$ of (4.4.3) is geometrically integral.*

*Proof.* Since $X$ is a complete intersection in $\mathbb{P}_K^{N+3}$, it is equidimensional and Cohen-Macaulay. As $X$ is Cohen-Macaulay, $X$ has no embedded components. The open subset $\{x_0 \neq 0\} \subset X$ is isomorphic to an open subset of the geometrically integral hypersurface $X' \subset \mathbb{P}_K^{N+2}$ in (4.4.4), see Lemma 4.4.2. Hence it suffices to show that the subset $\{x_0 = 0\} \subset X$ is not an irreducible component of $X$, which is clear because $f + a_0$ is an irreducible polynomial by (4.4.2). □

We turn to the special fibre of the family (4.4.3). The special fibre $Y = \mathcal{X} \times_R k$ of the family (4.4.3) has two components, namely

$$Y_0 \coloneqq \left\{ f + \sum_{i=0}^{l} a_i x_0^i y_j^i + x_0^{d-1} z = 0 \right\} \subset \mathbb{P}_k^{N+2}, \tag{4.4.7}$$

$$Y_1 \coloneqq \left\{ f + \sum_{i=0}^{l} a_i x_0^i y_j^i + x_0^{d-2}(\lambda y_j + x_0) w = 0 \right\} \subset \mathbb{P}_k^{N+2}. \tag{4.4.8}$$

The intersection $Z \coloneqq Y_0 \cap Y_1$ is the degree $d$ hypersurface

$$Z \coloneqq \left\{ f + \sum_{i=0}^{l} a_i x_0^i y_j^i = 0 \right\} \subset \mathbb{P}_k^{N+1}, \tag{4.4.9}$$

The assumption (see (4.4.2)) that $f + a_0$ is irreducible implies that $Z$ is integral.

**Lemma 4.4.5.** *The singular locus of $Y_0$ is contained in $\{x_0 = 0\}$ and the singular locus of $Y_1$ is contained in the closed subset $\{x_0(\lambda y_j + x_0) = 0\}$. Moreover,*

$$Y_1^{\text{sing}} \cap Z \subset Z^{\text{sing}}.$$

*Proof.* The derivative of the defining equation of $Y_0$ with respect to $z$ is given by

$$\partial_z \left( f + \sum_{i=0}^{l} a_i x_0^i y_j^i + x_0^{d-1} z \right) = x_0^{d-1}.$$

Hence, the singular locus of $Y_0$ is contained in $\{x_0 = 0\}$.

The derivative of the defining equation of $Y_1$ with respect to $w$ is given by

$$\partial_w \left( f + \sum_{i=0}^{l} a_i x_0^i y_j^i + x_0^{d-2}(\lambda y_j + x_0) w \right) = x_0^{d-2}(\lambda y_j + x_0).$$

Hence, the singular locus of $Y_1$ is contained in $\{x_0(\lambda y_j + x_0) = 0\}$ as claimed. Finally, the last claim in the lemma is clear, as $Z \subset Y_1$ is a Cartier divisor in $Y_1$ (given by $\{w = 0\} \subset Y_1$). □

In order to understand the obstruction map (4.3.1) for the family (4.4.3), we need to control the Chow group of the two components $Y_0$ and $Y_1$.

**Lemma 4.4.6.** *Let $Y_0$ and $Y_1$ be as in (4.4.7) and (4.4.8), i.e. the irreducible components of the special fibre of the family (4.4.3). Consider $D_0 \coloneqq \{x_0 = 0\} \subset Y_0$ and*

$D_1 \coloneqq \{x_0(\lambda y_j + x_0) = 0\} \subset Y_1$. *Then the natural push-forward maps*

$$\mathrm{CH}_1(D_0 \times_k L) \longrightarrow \mathrm{CH}_1(Y_0 \times_k L), \quad \mathrm{CH}_1(D_1 \times_k L) \longrightarrow \mathrm{CH}_1(Y_1 \times_k L)$$

*are surjective for every field extension $L/k$.*

*Proof.* Let $L/k$ be a field extension. Recall that

$$Y_0 \times_k L = \left\{ f + \sum_{i=0}^{l} a_i x_0^i y_j^i + x_0^{d-1} z = 0 \right\} \subset \mathbb{P}_L^{N+2},$$
$$Y_1 \times_k L = \left\{ f + \sum_{i=0}^{l} a_i x_0^i y_j^i + x_0^{d-2} (\lambda y_j + x_0) w = 0 \right\} \subset \mathbb{P}_L^{N+2}.$$

We consider $Y_1 \times_k L$. The projection away from $P = [0 : \cdots : 0 : 1] \in \mathbb{P}_L^{N+2}$ induces a rational map

$$\varphi \colon \mathbb{P}_L^{N+2} \dashrightarrow \mathbb{P}_L^{N+1}.$$

Since $w$ appears only linearly in the defining equation of $Y_1 \times_k L$, the restriction of $\varphi$ to $Y_1 \times_k L$ yields a birational map

$$Y_1 \times_k L \dashrightarrow \mathbb{P}_L^{N+1},$$

which induces an isomorphism between the complements of the closed subsets $D_1 \times_k L \subset Y_1 \times_k L$ and $H_1 \coloneqq \{x_0(\lambda y_j + x_0) = 0\} \subset \mathbb{P}_L^{N+1}$, respectively. Since $H_1$ is a union of two hyperplanes in $\mathbb{P}_L^{N+1}$, the pushforward along the natural inclusion

$$\mathrm{CH}_1(H_1) \longrightarrow \mathrm{CH}_1(\mathbb{P}_L^{N+1}) \simeq \mathbb{Z}$$

is surjective. Thus the localization exact sequence (see [Ful98, Proposition 1.8]) implies that

$$\mathrm{CH}_1(D_1 \times_k L) \longrightarrow \mathrm{CH}_1(Y_1 \times_k L)$$

is surjective, as $\mathrm{CH}_1(Y_1 \times_k L \setminus D_1 \times_k L) \simeq \mathrm{CH}_1(\mathbb{P}_L^{N+1} \setminus H_1) = 0$ by the above discussion. A similar argument shows that

$$\mathrm{CH}_1(D_0 \times_k L) \longrightarrow \mathrm{CH}_1(Y_0 \times_k L)$$

is surjective. This finishes the proof of the lemma. □

The following proposition is the main result of this section; it will be used for the induction step in our inductive argument.

**Proposition 4.4.7.** *Let $\Lambda$ be a ring of positive characteristic such that the exponential characteristic of $k_0$ is invertible in $\Lambda$. Let $\mathcal{X} \to \operatorname{Spec} R$ be the projective family from (4.4.3) with generic fibre $X$ for some $l \geq 2$. Let $Y_0$ and $Y_1$ be the irreducible components of the special fibre as in (4.4.7) and (4.4.8) and denote their scheme-theoretic intersection by $Z \coloneqq Y_0 \cap Y_1$. Let*

$$h \in R[x_0, \dots, x_n, y_1, \dots, y_{r+1}, z_1, \dots, z_s]$$

*be any homogeneous polynomial such that its reduction $h_0$ modulo the maximal ideal in $R$ has coefficients in $k_0$, i.e. $h_0 \in k_0[x_0, \dots, x_n, y_1, \dots, y_{r+1}, z_1, \dots, z_s]$ and such that $X$ is smooth over $K = k(t)$ away from $W_X \coloneqq \{x_0 h = 0\} \subset X$. Then*

$$\operatorname{Tor}^\Lambda(Z, W_Z) \mid \operatorname{Tor}^\Lambda(\bar{X}, W_{\bar{X}}),$$

*where $W_Z \coloneqq \{x_0 h_0 = 0\} \cup Z^{\text{sing}} \subset Z$ and $W_{\bar{X}} \coloneqq W_X \times_K \bar{K} \subset \bar{X} \coloneqq X \times_K \bar{K}$. In particular,*

$$\operatorname{Tor}^\Lambda(Z, W_Z) \mid \operatorname{Tor}^\Lambda(\bar{X}', W'),$$

*where $X'$ is as in Lemma 4.4.2, $\bar{X}' \coloneqq X' \times_K \bar{K}$, and $W' \coloneqq \{x_0 z_{s+1} h = 0\} \subset \bar{X}'$.*

**Remark 4.4.8.** As a consequence of Lemma 4.4.1, the smoothness of $X \setminus W_X$ is automatically satisfied in characteristic 0. This is not true in general, but we will choose in our applications $W_X$ carefully so that the condition holds (over any field).

*Proof.* Since $\Lambda$ is a ring of positive characteristic,

$$m \coloneqq \operatorname{Tor}^\Lambda(\bar{X}, W_{\bar{X}})$$

is a positive integer $m \in \mathbb{Z}_{\geq 1}$.

**Step 1.** We will check that the assumptions in Theorem 4.3.3 are satisfied for the projective family $\mathcal{X} \to \operatorname{Spec} R$ from (4.4.3) with the closed subset

$$W_{\mathcal{X}} \coloneqq \{x_0 h = 0\} \cup Y_1^{\text{sing}} \cup Z^{\text{sing}} \subset \mathcal{X}.$$

As in Theorem 4.3.3, let $W_Y \coloneqq W_{\mathcal{X}} \cap Y$. We note that $W_X$ agrees with $W_{\mathcal{X}} \cap X$. The generic fibre $X$ of $\mathcal{X} \to \operatorname{Spec} R$ is geometrically integral by Corollary 4.4.4. The special fibre $Y^\circ \coloneqq Y \setminus W_Y$ consists of two components $Y^\circ = Y_0^\circ \cup Y_1^\circ$ such that $Y_0^\circ$, $Y_1^\circ$, and their intersection $Z^\circ \coloneqq Y_0^\circ \cap Y_1^\circ$ are smooth and integral, see also Lemma

4.4.5. In particular, $Y^\circ$ is an snc scheme, see Definition 2.3.1. The singular locus of $\mathcal{X}$ is contained in $\{x_0 = 0\}$ by Lemma 4.4.1. It follows from this that $Y_i^\circ$ is a Cartier divisor on $\mathcal{X}^\circ$ for $i = 0, 1$. The generic fibre of the $R$-scheme $\mathcal{X}^\circ := \mathcal{X} \setminus W_{\mathcal{X}}$ is equal to $X \setminus W_X$ and thus smooth by assumption. In particular, $\mathcal{X}^\circ \to \operatorname{Spec} R$ is strictly semi-stable, see Definition 2.3.2. It follows that the assumptions (1) and (2) in Theorem 4.3.3 are satisfied for $\mathcal{X} \to \operatorname{Spec} R$ and the closed subset $W_{\mathcal{X}} \subset \mathcal{X}$. This concludes Step 1.

Recall that $Z$ is integral and let $\delta_{Z^\circ} \in Z^\circ_{k(Z)}$ denote the diagonal point of $Z^\circ$, which is dense open in $Z$.

**Step 2.** We will show that there is a one-cycle $\gamma \in \mathrm{CH}_1(Y_1 \times_k k(Z), \Lambda)$, supported on $\{x_0(\lambda y_j + x_0) = 0\} \subset Y_1 \times_k k(Z)$, such that

$$m \cdot \delta_{Z^\circ} = (\iota^*\gamma)|_{Z^\circ_{k(Z)}} \in \mathrm{CH}_0(Z^\circ_{k(Z)}, \Lambda), \tag{4.4.10}$$

where $\iota\colon Y_1^\circ \times_k k(Z) \hookrightarrow Y_1 \times_k k(Z)$ denotes the natural open embedding and $(\iota^*\gamma)|_{Z^\circ_{k(Z)}}$ the pullback of the one-cycle to the Cartier divisor $Z^\circ_{k(Z)} \subset Y_1^\circ \times_k k(Z)$ along the natural regular embedding.

By Step 1, Theorem 4.3.3 implies that the cokernel of the map

$$\Psi^\Lambda_{Y_L^\circ}\colon\ \mathrm{CH}_1(Y_0^\circ \times_k L, \Lambda) \oplus \mathrm{CH}_1(Y_1^\circ \times_k L, \Lambda) \longrightarrow \mathrm{CH}_0(Z^\circ \times_k L, \Lambda)$$

from (4.3.1) (with $0 < 1$) is $m$-torsion for every field extension $L/k$. In particular,

$$m \cdot \delta_{Z^\circ} \in \operatorname{im} \Psi^\Lambda_{Y^\circ_{k(Z)}}.$$

Note that $\mathrm{CH}_1(Y_0^\circ \times_k k(Z), \Lambda) = 0$ by Lemma 4.4.6 and that the pull-back

$$\iota^*\colon\ \mathrm{CH}_1(Y_1 \times_k k(Z), \Lambda) \longrightarrow \mathrm{CH}_1(Y_1^\circ \times_k k(Z), \Lambda)$$

is surjective by [Ful98, Proposition 1.8]. Hence there exists a one-cycle $\gamma \in \mathrm{CH}_1(Y_1 \times_k k(Z), \Lambda)$ such that $m \cdot \delta_{Z^\circ} = (\iota^*\gamma)|_{Z^\circ_{k(Z)}}$. By Lemma 4.4.6 we can further assume that $\gamma$ is supported on $\{x_0(\lambda y_j + x_0) = 0\} \subset Y_1 \times_k k(Z)$. This concludes Step 2.

**Step 3.** We specialize now $\lambda \to 0$ and aim to compute the image of (4.4.10) under the corresponding specialization map on Chow groups (see Lemma 2.2.1); we will show that the specialization of the one-cycle $\gamma$ vanishes and so does the specialization of $m \cdot \delta_{Z^\circ}$.

Consider the discrete valuation ring $B = k_0[\lambda]_{(\lambda)}$ with residue field $k_0$ and fraction field $k_0(\lambda)$. Recall that $k$ is an algebraic closure of $k_0(\lambda)$. Then consider the flat projective $B$-schemes

$$\mathcal{Z} := \left\{ f + \sum_{i=0}^{l} a_i x_0^i y_j^i = 0 \right\} \subset \mathbb{P}_B^{N+1},$$
$$\mathcal{Y}_1 := \left\{ f + \sum_{i=0}^{l} a_i x_0^i y_j^i + x_0^{d-2}(\lambda y_j + x_0) w = 0 \right\} \subset \mathbb{P}_B^{N+2},$$

where $f$ and $a_i$ are as in (4.4.1). Note that $Z$ and $Y_1$ are the geometric generic fibre of $\mathcal{Z}$ and $\mathcal{Y}_1$, respectively. Let $\mathcal{Z}^\circ \subset \mathcal{Z}$ and $\mathcal{Y}_1^\circ \subset \mathcal{Y}_1$ be the complement of the closure of $W_Z$ in $\mathcal{Z}$ and of $W_{Y_1}$ in $\mathcal{Y}_1$, respectively. Note that $Z^\circ = Z \backslash W_Z$ and $Y_1^\circ = Y_1 \backslash W_{Y_1}$ are the geometric generic fibres of $\mathcal{Z}^\circ$ and $\mathcal{Y}_1^\circ$, respectively. We denote the special fibres by $Z_0^\circ$ and $Y_{1,0}^\circ$, respectively. By Lemma 2.2.1, there exist specialization maps induced by Fulton's specialization map for the flat $\mathcal{O}_{\mathcal{Z},Z_0}$-schemes $\mathcal{Z}^\circ \times_B \mathcal{O}_{\mathcal{Z},Z_0}$, $\mathcal{Y}_1^\circ \times_B \mathcal{O}_{\mathcal{Z},Z_0}$, and $\mathcal{Y}_1 \times_B \mathcal{O}_{\mathcal{Z},Z_0}$

$$\begin{aligned}
\mathrm{sp}_{Z^\circ} \colon \mathrm{CH}_0(Z^\circ \times_k k(Z), \Lambda) &\longrightarrow \mathrm{CH}_0(Z_0^\circ \times_{k_0} k_0(Z_0), \Lambda),\\
\mathrm{sp}_{Y_1^\circ} \colon \mathrm{CH}_1(Y_1^\circ \times_k k(Z), \Lambda) &\longrightarrow \mathrm{CH}_1(Y_{1,0}^\circ \times_{k_0} k_0(Z_0), \Lambda),\\
\mathrm{sp}_{Y_1} \colon \mathrm{CH}_1(Y_1 \times_k k(Z), \Lambda) &\longrightarrow \mathrm{CH}_1(Y_{1,0} \times_{k_0} k_0(Z_0), \Lambda),
\end{aligned}$$

where $Y_{1,0}$ denotes the special fibre of the $B$-scheme $\mathcal{Y}_1$.

We apply now the specialization $\mathrm{sp}_{Z^\circ}$ to the zero-cycle (4.4.10). By Lemma 2.2.1, we get

$$m \cdot \delta_{Z_0^\circ} = \mathrm{sp}_{Z^\circ}(m \cdot \delta_{Z^\circ}) = \mathrm{sp}_{Z^\circ}((\iota^*\gamma)|_{Z^\circ_{k(Z)}}) = \mathrm{sp}_{Y_1^\circ}(\iota^*\gamma)|_{Z^\circ_{0,k_0(Z_0)}}, \tag{4.4.11}$$

where $|_{Z^\circ_{k(Z)}}$ and $|_{Z^\circ_{0,k_0(Z_0)}}$ denote the pullback along the regular embedding of the Cartier divisor $Z^\circ_{k(Z)}$ and $Z^\circ_{0,k_0(Z_0)}$ in $Y_1^\circ \times_k k(Z)$ and $Y_{1,0}^\circ \times_{k_0} k_0(Z_0)$, respectively. Recall that the one-cycle $\gamma$ is supported on $\{x_0(\lambda y_j + x_0) = 0\} \subset Y_1 \times_k k(Z)$. Thus the specialization $\mathrm{sp}_{Y_1}(\gamma) \in \mathrm{CH}_1(Y_{1,0}, \Lambda)$ is supported on $\{x_0^2 = 0\} \subset Y_{1,0}$. In particular

$$\mathrm{sp}_{Y_1^\circ}(\iota^*\gamma) = 0 \in \mathrm{CH}_1(Y_{1,0}^\circ),$$

as the subset $\{x_0 = 0\} \subset Y_{1,0}$ is contained in the specialization of $W_{Y_1}$ and sp commutes with pullbacks along open immersions. Hence, the right hand side of (4.4.11) vanishes, which concludes Step 3. By Step 3,

$$\mathrm{Tor}^\Lambda(Z_0, W_{Z_0}) \mid m,$$

where $W_{Z_0} \subset Z_0$ is the specialization of $W_Z \subset Z$. We note that $Z = Z_0 \times_{k_0} k$ and $W_Z = W_{Z_0} \times_{k_0} k$, because the defining equations of $Z$ and $W_Z$ are defined over $k_0$. Hence, the proposition follows from Lemma 4.2.6 (e), as $m = \mathrm{Tor}^{\Lambda}(\bar{X}, W_{\bar{X}})$. $\square$

## 4.5. Base case

Our argument will rely on an inductive application of a degeneration as in Section 4.4. For the start of the induction we will use the explicit example of a singular hypersurface with nontrivial unramified cohomology from the proof of [Sch21a, Theorem 7.1]. We recall the example in what follows.

Let $k$ be an algebraically closed field and let $m \geq 2$ be an integer coprime to the exponential characteristic of $k$. Let $n \geq 2$ and $r \leq 2^n - 2$ be positive integers. Let $x_0, \dots, x_n, y_1, \dots, y_{r+1}$ be the coordinates of $\mathbb{P}^{n+r+1}$ and let $\pi \in k$ be an element that is transcendental over the prime field of $k$. Consider the homogeneous polynomial

$$g(x_0, \dots, x_n) \coloneqq \pi \cdot \left( \sum_{i=0}^{n} x_i^{\left\lceil \frac{n+1}{m} \right\rceil} \right)^m - (-1)^n x_0^{m\left\lceil \frac{n+1}{m} \right\rceil - n} x_1 x_2 \cdots x_n \tag{4.5.1}$$

in $k[x_0, \dots, x_n]$ of degree $\deg g = m \left\lceil \frac{n+1}{m} \right\rceil \leq n + m$, see [Sch21a, Equation (21)]. Using this we define the homogeneous polynomial

$$F \coloneqq g(x_0, \dots, x_n) x_0^{m+n-\deg(g)} + \sum_{j=1}^{r} x_0^{n - \deg c_j} c_j(x_1, \dots, x_n) y_j^m + (-1)^n x_1 x_2 \dots x_n y_{r+1}^m$$

in $k[x_0, \dots, x_n, y_1, \dots, y_{r+1}]$ of degree $m + n$, where

$$c_j(x_1, \dots, x_n) \coloneqq (-x_1)^{\varepsilon_1} (-x_2)^{\varepsilon_2} \cdots (-x_n)^{\varepsilon_n} \in k[x_1, \dots, x_n] \tag{4.5.2}$$

with $\varepsilon_i$ the $(i-1)$-th digit in the 2-adic representation of $j$, i.e. $j = \sum_i \varepsilon_i 2^{i-1}$ with $\varepsilon_i \in \{0, 1\}$. Consider the associated degree $m + n$ hypersurface

$$Z \coloneqq \{F = 0\} \subset \mathbb{P}_k^{n+r+1}. \tag{4.5.3}$$

**Theorem 4.5.1.** *Let $l \in k[x_0, \dots, x_n]$ be any nontrivial homogeneous polynomial and consider*

$$W_Z \coloneqq \{y_{r+1} l = 0\} \cup Z^{\mathrm{sing}} \subset Z$$

*where $Z$ is as in (4.5.3). Then $\mathrm{Tor}^{\mathbb{Z}/m}(Z, W_Z) = m$.*

We have the following immediate corollary.

**Corollary 4.5.2.** *Let $h, l', l'' \in k[x_0, \dots, x_n]$ be nontrivial homogeneous polynomials such that $h$ is irreducible of degree $m + n + \deg l'$. Let $k' = k(\rho)$ be a purely transcendental field extension of $k$. Consider the hypersurface*

$$Z_\rho \coloneqq \{\rho h + l' \cdot F = 0\} \subset \mathbb{P}^{n+r+1}_{k'}$$

*of degree $m + n + \deg l'$. Then $\operatorname{Tor}^{\mathbb{Z}/m}(\bar{Z}_\rho, W_{\bar{Z}_\rho}) = m$, where $\bar{Z}_\rho = Z_\rho \times_{k'} \bar{k}'$ and*

$$W_{\bar{Z}_\rho} \coloneqq \{y_{r+1} l' l'' = 0\} \cup Z_\rho^{\text{sing}} \times_{k'} \bar{k}' \subset \bar{Z}_\rho.$$

*Proof.* Consider the pair $(Z_\rho, W_{Z_\rho})$ and let $(Z_0, W_{Z_0})$ be the pair obtained by specializing $\rho \to 0$, i.e.

$$Z_0 = \{l' \cdot F = 0\} \subset \mathbb{P}^{n+r+1}_k,$$
$$W_{Z_0} = \{y_{r+1} l' l'' = 0\} \cup Z_0^{\text{sing}} \subset Z_0.$$

Note that the scheme $Z_0$ is reducible, but the open subscheme $U_0 \coloneqq Z_0 \backslash W_{Z_0} \subset Z_0$ is integral as the polynomial $F$ is irreducible. Hence, the torsion order $\operatorname{Tor}^{\mathbb{Z}/m}(Z_0, W_{Z_0})$ is defined, see Remark 4.2.5. We observe that $U_0 = Z \setminus W_Z$, where the pair $(Z, W_Z)$ is as in Theorem 4.5.1 with $l = l' \cdot l'' \in k[x_0, \dots, x_n]$. Thus, we get $\operatorname{Tor}^{\mathbb{Z}/m}(Z_0, W_{Z_0}) = m$ by Theorem 4.5.1. Hence, $m$ divides $\operatorname{Tor}^{\mathbb{Z}/m}(\bar{Z}_\rho, W_{\bar{Z}_\rho})$ by Lemma 4.2.9. Conversely, any $\mathbb{Z}/m$-torsion order can be at most $m$. Hence, $\operatorname{Tor}^{\mathbb{Z}/m}(\bar{Z}_\rho, W_{\bar{Z}_\rho}) = m$, as claimed. □

*Proof of Theorem 4.5.1.* This follows from arguments similar to those in the proof of [Sch21a, Theorem 6.1 and 7.1]; we give some details for the reader's convenience. Let $P = \{x_0 = \cdots = x_n = 0\} \subset \mathbb{P}^{n+r+1}_k$ and consider the blow-up $Y \coloneqq Bl_P Z$, which can be described via the vanishing locus of

$$g x_0^{m+n-\deg(g)} y_0^m + \sum_{j=1}^{r} x_0^{n - \deg c_j} c_j(x_1, \dots, x_n) y_j^m + (-1)^n x_1 x_2 \dots x_n y_{r+1}^m, \quad (4.5.4)$$

inside the projective bundle $\mathbb{P}_{\mathbb{P}^n}(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus(r+1)})$ over $\mathbb{P}^n$, where $y_0$ denotes a local coordinate that trivializes $\mathcal{O}(-1)$ and $y_1, \dots, y_{r+1}$ trivialize $\mathcal{O}^{\oplus(r+1)}$. The projection to the $x$-coordinates induces a morphism $f : Y \to \mathbb{P}^n_k$. We furthermore pick an alteration $\tau' : Y' \to Y$ of order coprime to $m$, which can be done by [Tem17], see also [IT14]. The corresponding alteration of $Z$ is denoted by $\tau : Y' \to Z$.

As detailed in [Sch21a, §7], an application of [Sch21a, Theorem 5.3] shows that the pullback of the class $\alpha \coloneqq (x_1/x_0, \ldots, x_n/x_0) \in H^n(k(\mathbb{P}^n), \mu_m^{\otimes n})$ yields an unramified class

$$\gamma \coloneqq f^*\alpha \in H^n_{nr}(k(Y)/k, \mu_m^{\otimes n}) = H^n_{nr}(k(Z)/k, \mu_m^{\otimes n})$$

of order $m$ such that for any subvariety $E \subset Y'$ which does not dominate $\mathbb{P}^n$ via $f \circ \tau'$, the class $(\tau')^*\gamma$ vanishes in $H^n(k(E), \mu_m^{\otimes n})$. Moreover, the class $f^*\alpha$ vanishes at the generic point of the exceptional divisor $D_0$ of $Y \to Z$, which is cut out by $y_0 = 0$, and at the generic point of the strict transform $D_{r+1}$ of the divisor $\{y_{r+1} = 0\} \subset Z$ under the blow-up morphism $Y \to Z$. Indeed, the divisors $D_0$ and $D_{r+1}$ in $Y$ map via $f$ onto $\mathbb{P}^n$ and the generic fibres $D_{0,\eta}$ and $D_{r+1,\eta}$ of $f|_{D_i}$ are the hypersurfaces in (different) $\mathbb{P}^r_{k(\mathbb{P}^n)}$ given by

$$\sum_{j=1}^{r} c_j(x_1, \ldots, x_n) y_j^m + (-1)^n x_1 x_2 \ldots x_n y_{r+1}^m = 0 \quad \text{and}$$
$$\pi \cdot \left(1 + \sum_{i=1}^{n} x_i^{\lceil \frac{n+1}{m} \rceil}\right)^m y_0^m - (-1)^n x_1 x_2 \cdots x_n y_0^m + \sum_{j=1}^{r} c_j(x_1, \ldots, x_n) y_j^m = 0,$$

respectively, see (4.5.4) and (4.5.1). Thus $D_{0,\eta}$ and $D_{r+1,\eta}$ are each isomorphic as $k(\mathbb{P}^n)$-varieties to a subvariety of the hypersurface given by the vanishing of the $n$-th Fermat-Pfister form of degree $m$

$$\sum_{\varepsilon \in \{0,1\}^n} (-x_1)^{\varepsilon_1} \ldots (-x_n)^{\varepsilon_n} y_{j(\varepsilon)}^m$$

with $j(\varepsilon) = \sum_{i=0}^{n} \varepsilon_i 2^{i-1}$ as defined in [Sch21a, Equation (7)]. The vanishing of $f^*\alpha$ along the generic points of $D_0$ and $D_{r+1}$ follows from [Sch21a, Corollary 4.3]. Since the generic fibre of $f$ is smooth ($m$ is invertible in $k$), we conclude via [BO74] that

$$((\tau')^*\gamma)|_E = 0 \in H^n(k(E), \mu_m^{\otimes n}) \tag{4.5.5}$$

for any subvariety $E \subset Y'$ with $\tau(E) \subset W_Z = Z^{\text{sing}} \cup \{y_{r+1} l = 0\}$, where $l \in k[x_0, \ldots, x_n]$ is any nontrivial homogeneous polynomial in $x_0, \ldots, x_n$ as in the statement of the theorem.

A computation with the Merkurjev pairing similar to [Sch19b, §3] or [Sch21b, Theorem 8.6] then shows $\mathrm{Tor}^{\mathbb{Z}/m}(Z, W_Z) = m$; we sketch the argument for convenience. For a contradiction assume that there is a positive integer $m' < m$ with

$$m' \cdot \delta_Z = z + m \cdot \zeta \in \mathrm{CH}_0(Z_{k(Z)})$$

for some zero-cycle $z$ whose support $\operatorname{supp} z$ lies in $(W_Z)_{k(Z)}$ and some zero-cycle $\zeta \in \mathrm{CH}_0(Z_{k(Z)})$ that reflects the fact that we work with $\mathbb{Z}/m$-torsion orders. We restrict the above identity to the regular locus of $Z_{k(Z)}$ and pull this pack to $\tau^{-1}(Z^{\mathrm{sm}}_{k(Z)})$. The localization sequence then yields

$$m' \cdot \delta_\tau = z' + m \cdot \zeta' \in \mathrm{CH}_0(Y'_{k(Z)}),$$

where $\delta_\tau = \tau^*\delta_Z$ is the point induced by the graph of $\tau : Y' \to Z$, $z'$ is a zero-cycle with $\operatorname{supp} z' \subset \tau^{-1}(W_Z)_{k(Z)}$ and $\zeta' \in \mathrm{CH}_0(Y'_{k(Z)})$. Note that this used $Z^{\mathrm{sing}} \subset W_Z$. We pair the above zero-cycle via the Merkurjev pairing (see [Mer08, §2.4] or [Sch21b, §5]) with the unramified class $\gamma$ from above. This yields

$$\langle m' \cdot \delta_\tau, \tau^*\gamma \rangle = 0,$$

because $\gamma$ is $m$-torsion, hence pairs to zero with $m \cdot \zeta'$, and it restricts to zero on generic points of subvarieties of $\tau^{-1}(W_Z)$ by (4.5.5), hence pairs to zero with $z'$. Conversely, the definition of the Merkurjev pairing directly implies that

$$0 = \langle m' \cdot \delta_\tau, \tau^*\gamma \rangle = m' \cdot \tau_*\tau^*\gamma = m' \cdot \deg(\tau) \cdot \gamma \in H^n(k(Z), \mu_m^{\otimes n}),$$

as $\delta_\tau \in Y'_{k(Z)}$ is the point associated to the graph $\Gamma_\tau \subset Y' \times Z$ of $\tau$. This contradicts the fact that $\deg(\tau)$ is coprime to $m$, that $1 \leq m' < m$, and that $\gamma$ has order $m$. This concludes the proof of the theorem. □

## 4.6. Proof of the main results

Let $k$ be an algebraically closed field and let $X \subset \mathbb{P}^{N+1}_k$ be a smooth Fano hypersurface, i.e. a smooth hypersurface of degree $d \leq N+1$. Then $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ is an isomorphism and so the torsion order $\mathrm{Tor}(X)$ of $X$ is the torsion order $\mathrm{Tor}^{\mathbb{Z}}(X, W)$ of $X$ relative to any closed zero-dimensional subset $W \subset X$, see Lemma 4.2.6 (d). The main results of this chapter, stated in the introduction, will follow from the following result.

**Theorem 4.6.1.** *Let $k$ be a field and let $m \geq 2$ be an integer invertible in $k$. Let $n \geq 2$, $r \leq 2^n - 2$, and let*

$$s \leq \sum_{l=1}^{n} \binom{n}{l} \left\lfloor \frac{n-l}{m} \right\rfloor$$

*be non-negative integers. Write $N := n + r + s$. Then the torsion order $\mathrm{Tor}(X_d)$ of* *a very general Fano hypersurface $X_d \subset \mathbb{P}_k^{N+1}$ of degree $d \geq m+n$ is divisible by $m$.*

**Remark 4.6.2.** For $s=0$, the result is proven in [Sch21a, Theorem 7.1].

*Proof of Theorem 4.6.1.* Note that the torsion order of any variety is divisible by the torsion order of the base-change to any field extension. Moreover, the definition of very general (see Definition 2.4.1) is stable under extension of the base field. Up to replacing $k$ by a field extension, we can therefore assume that $k$ is algebraically closed and uncountable.

We fix positive integers $n \geq 2$ and $r \leq 2^n - 2$. For a non-negative integer $s$ as in the theorem, we define inductively an integral degree $d$-hypersurface $Z = Z_s$ of dimension $N = n + r + s$.

**Step 1.** Suppose there exists an integral degree $d$-hypersurface $Z_s \subset \mathbb{P}_k^{n+r+s+1}$ given by the vanishing of a homogeneous polynomial of the form

$$f_0^{(s)} + a_0^{(s)} + a_{r+1}^{(s)} y_{r+1}^m + \sum_{j=1}^{r} \sum_{i=1}^{m} a_{i,j}^{(s)} \cdot y_j^i \in k[x_0, \ldots, x_n, y_1, \ldots, y_{r+1}, z_1, \ldots, z_s] \quad (4.6.1)$$

for some homogeneous polynomials

$$f_0^{(s)}, a_0^{(s)}, a_{r+1}^{(s)}, a_{i,j}^{(s)} \in k[x_0, \ldots, x_n, z_1, \ldots, z_s]$$

such that

(1) $f_0^{(s)} + a_0^{(s)} \in k[x_0, \ldots, x_n, z_1, \ldots, z_s]$ is an irreducible polynomial of degree $d$;

(2) for each $j$, if $x_0^{em} \mid a_{m,j}^{(s)}$ for a non-negative integer $e$, then $x_0^{ei} \mid a_{i,j}^{(s)}$ for all $1 \leq i \leq m$. We denote the maximal such $e$ by $e_j^{(s)}$;

(3) $\operatorname{Tor}^{\mathbb{Z}/m}(Z_s, W_s) = m$, where $W_s := \{x_0 a_{r+1}^{(s)} y_{r+1} h^{(s)} = 0\} \cup Z_s^{\mathrm{sing}} \subset Z_s$ for some homogeneous polynomial $h^{(s)} \in k[x_0, \ldots, x_n, z_1, \ldots, z_s]$.

Assume that there exists some $1 \leq j_0 \leq r$ such that $e_{j_0}^{(s)} \geq 1$. Then we construct an integral degree $d$-hypersurface $Z_{s+1}$ of the same form (4.6.1) satisfying the condition (1), (2), and (3) as follows.

Consider $k' = \overline{k(\lambda)}$ an algebraic closure of the purely transcendental field extension $k(\lambda)$ of $k$. Rewrite the equation (4.6.1) as

$$\left( f_0^{(s)} + a_{r+1}^{(s)} y_{r+1}^m + \sum_{j \neq j_0} \sum_{i=1}^{m} a_{i,j}^{(s)} \cdot y_j^i \right) + a_0^{(s)} + \sum_{i=1}^{m} a_{i,j_0}^{(s)} y_{j_0}^i .$$

We are now in the situation of Section 4.4. Note that condition (1) implies the assumption (4.4.2). Let $Z_{s+1}$ be the integral degree $d$-hypersurface $X' \times_K \bar{K} \subset \mathbb{P}^{n+r+s+2}_{\bar{K}}$, where $X'$ is as in Lemma 4.4.2 and $K = k'(t)$. We choose an isomorphism $k \simeq \bar{K}$, which exists because both fields are algebraically closed, have the same characteristic and have the same uncountable transcendence degree over their prime fields. Thus we can view $Z_{s+1}$ as a variety over $k$. We aim to check that $Z_{s+1}$ satisfies the assumptions above. Recall from Lemma 4.4.2 that $Z_{s+1} \subset \mathbb{P}^{n+r+s+2}_k$ is cut out by the homogeneous polynomial

$$\left( f_0^{(s)} + a_{r+1}^{(s)} y_{r+1}^m + \sum_{j \neq j_0} \sum_{i=1}^{m} a_{i,j}^{(s)} \cdot y_j^i \right) + a_0^{(s+1)} + \sum_{i=1}^{m} a_{i,j_0}^{(s+1)} y_{j_0}^i, \tag{4.6.2}$$

where $a_0^{(s+1)}, a_{i,j_0}^{(s+1)} \in k[x_0, \ldots, x_n, z_1, \ldots, z_{s+1}]$ are defined as in (4.4.6). Note this uses also Lemma 4.4.2 (3). Hence, the defining equation of $Z_{s+1}$ has the form (4.6.1) with

$$f_0^{(s+1)} := f_0^{(s)}, \quad a_{r+1}^{(s+1)} := a_{r+1}^{(s)}, \quad \text{and} \quad a_{i,j}^{(s+1)} := a_{i,j}^{(s)} \quad \text{for } j \neq j_0.$$

Since $f_0^{(s+1)}$ and $a_0^{(s+1)}$ do not contain any $y_i$, condition (1) follows from Lemma 4.4.2 (4). As $a_{i,j}^{(s+1)} = a_{i,j}^{(s)}$ for $j \neq j_0$, condition (2) is clearly satisfied for $j \neq j_0$ and we note that $e_j^{(s+1)} = e_j^{(s)}$ for $j \neq j_0$. For $j = j_0$, condition (2) follows from Lemma 4.4.2 (2). Proposition 4.4.7 shows that $\mathrm{Tor}^{\mathbb{Z}/m}(Z_{s+1}, W_{s+1}) = m$, where $W_{s+1} = \{x_0 a_{r+1}^{(s)} y_{r+1} h^{(s)} z_{s+1} = 0\} \subset Z_{s+1}$. Note that the derivative of (4.6.2) with respect to $y_{r+1}$ is equal to $m a_{r+1}^{(s)} y_{r+1}^{m-1}$; thus the singular locus of $Z_{s+1}$ is contained in $\{a_{r+1}^{(s)} y_{r+1} = 0\} \subset Z_{s+1}$. By definition, $a_{r+1}^{(s+1)} = a_{r+1}^{(s)}$ and so condition (3) is satisfied for $h^{(s+1)} = h^{(s)} z_{s+1}$.

**Step 2.** Consider the hypersurface $Z_0 := Z_\rho$ with defining equation $\rho h + x_0^{d-m-n} F$ as in Corollary 4.5.2, where $h \in k[x_0, \ldots, x_n]$ is an irreducible polynomial of degree $d$, e.g.

$$h = \begin{cases} x_0^d + \sum\limits_{i=1}^{n} x_{i-1} x_i^{d-1} & \text{if } p > 0 \text{ and } p \mid d, \\ \sum\limits_{i=0}^{n} x_i^d & \text{otherwise,} \end{cases}$$

where $p$ is the characteristic of $k$. We will prove that $Z_0$ satisfies the condition (1), (2), and (3) above. Consider the following polynomials in $k[x_0, \ldots, x_n]$

$$f_0^{(0)} := \rho h + x_0^{d-\deg(g)} g, \quad a_0^{(0)} := 0, \quad a_{r+1}^{(0)} := (-1)^n x_0^{d-m-n} x_1 x_2 \ldots x_n,$$
$$a_{i,j}^{(0)} := 0, \quad \text{for } 1 \le i \le m-1 \text{ and } 1 \le j \le r,$$

$$a_{m,j}^{(0)} := x_0^{d-m-\deg(c_j)} c_j(x_1, \dots, x_n), \quad \text{for } 1 \le j \le r,$$

where $g$ is defined in (4.5.1) and the $c_j$'s are defined in (4.5.2). Then, by construction,

$$\rho h + x_0^{d-m-n} F = f_0^{(0)} + a_0^{(0)} + a_{r+1}^{(s)} y_{r+1}^m + \sum_{j=1}^{r} \sum_{i=1}^{m} a_{i,j}^{(0)} \cdot y_j^i \in k[x_0, \dots, x_n, y_1, \dots, y_{r+1}]$$

is of the form (4.6.1). The polynomial $f_0^{(0)}$ is irreducible because the polynomial $h$ is irreducible and $\rho$ is a transcendental parameter over the prime field of $k$, which is algebraically independent from $\pi$. Hence condition (1) holds. Condition (2) is clearly satisfied as $a_{i,j}^{(0)} = 0$ for $1 \le i \le m-1$. In particular, we see from the definition of $a_{m,j}^{(0)}$ above that

$$e_j^{(0)} = \left\lfloor \frac{d - m - \deg(c_j)}{m} \right\rfloor. \tag{4.6.3}$$

Corollary 4.5.2 shows that condition (3) holds as well, where we note that we can choose $h^{(0)} = 1$. This concludes Step 2.

By Steps 1 and 2 above, we can apply the double cone construction (see Section 4.4) as long as at least one of the $e_j$'s defined in (2) is positive. In each step, we reduce one of them by 1. Hence, the number of steps is equal to the sum

$$\sum_{j=1}^{r} e_j^{(0)} = \sum_{j=1}^{r} \left\lfloor \frac{d - m - \deg(c_j)}{m} \right\rfloor.$$

This sum becomes maximal when $r$ is maximal, i.e. $r = 2^n - 2$. Then the sum reads

$$\sum_{j=1}^{2^n-2} e_j^{(0)} = \sum_{l=1}^{n-1} \binom{n}{l} \left\lfloor \frac{d-m-l}{m} \right\rfloor.$$

Let now $X_d$ be a very general hypersurface of degree $d$ and dimension $N$ over $k$. Up to replacing $k$ by a larger algebraically closed field (which does not affect the torsion order by Lemma 4.2.6), we can by Lemma 2.4.3 assume that $X_d$ degenerates to $Z_s$. The intersection of the hyperplane section $\{x_0 = 0\}$ with $N-1$ general hyperplane sections is a closed subset $W$ in the total space of the degeneration which has relative dimension 0 and whose restriction to the special fibre $Z_s$ is contained in $W_s$. Applying Lemma 4.2.9 to the degeneration with $W$ as closed subset yields that

$$\mathrm{Tor}^{\mathbb{Z}/m}(X_d, W_{X_d}) = m,$$

where $W_{X_d} := W \cap X_d \subset X_d$ is a closed non-empty zero-dimensional subset of $X_d$. Thus, Lemma 4.2.6 implies that $m$ divides $\operatorname{Tor}(X_d)$, which concludes the proof. □

The following lemmas yield explicit estimates for the bound given in Theorem 4.6.1.

**Lemma 4.6.3.** *Let $n, m \geq 2$ be positive integers. Then*

$$\left(\left\lfloor \frac{n}{m} \right\rfloor - 1\right)(2^{n-1}-1) \leq \sum_{l=1}^{n} \binom{n}{l} \left\lfloor \frac{n-l}{m} \right\rfloor \leq \left\lfloor \frac{n}{m} \right\rfloor (2^{n-1}-1). \tag{4.6.4}$$

*Proof.* The sum in question can be rewritten as follows

$$\begin{aligned} \sum_{l=1}^{n} \binom{n}{l} \left\lfloor \frac{n-l}{m} \right\rfloor &= \frac{1}{2}\sum_{l=1}^{n-1} \binom{n}{l} \left\lfloor \frac{n-l}{m} \right\rfloor + \frac{1}{2}\sum_{l=1}^{n-1} \binom{n}{n-l} \left\lfloor \frac{n-l}{m} \right\rfloor \\ &= \frac{1}{2}\sum_{l=1}^{n-1} \binom{n}{l} \left\lfloor \frac{n-l}{m} \right\rfloor + \frac{1}{2}\sum_{l=1}^{n-1} \binom{n}{l} \left\lfloor \frac{l}{m} \right\rfloor \\ &= \frac{1}{2}\sum_{l=1}^{n-1} \binom{n}{l} \left(\left\lfloor \frac{n-l}{m} \right\rfloor + \left\lfloor \frac{l}{m} \right\rfloor\right). \end{aligned}$$

The estimates in (4.6.4) follow now from the observation

$$\left(\left\lfloor \frac{n}{m} \right\rfloor - 1\right) \leq \left(\left\lfloor \frac{n-l}{m} \right\rfloor + \left\lfloor \frac{l}{m} \right\rfloor\right) \leq \left\lfloor \frac{n}{m} \right\rfloor$$

for all $0 \leq l \leq n$ together with the summation formula for binomial coefficients. □

The more explicit formulas for $m = 2$ and $m = 3$ below are used in Theorem 4.1.1 and Theorem 4.1.2.

**Lemma 4.6.4.** *Let $n$ be a positive integer. Then the following formulas hold*

$$\sum_{l=1}^{n} \binom{n}{l} \left\lfloor \frac{n-l}{2} \right\rfloor = (n-1)2^{n-2} - \left\lfloor \frac{n}{2} \right\rfloor, \tag{4.6.5}$$

$$\sum_{l=1}^{n} \binom{n}{l} \left\lfloor \frac{n-l}{3} \right\rfloor = \frac{n-2}{3}2^{n-1} - \frac{n}{3} + \delta, \tag{4.6.6}$$

*where $\delta$ depends on the remainder of $n$ modulo 6 and is given by the following table.*

| $n \mod 6$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\delta$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | 0 | $\frac{2}{3}$. |

*Proof.* We check (4.6.5) by the following computation:

$$\begin{aligned}
\sum_{l=1}^{n} \binom{n}{l} \left\lfloor \frac{n-l}{2} \right\rfloor &= \sum_{l=1}^{n} \frac{n-l}{2} \binom{n}{l} - \frac{1}{2} \sum_{\substack{l=1 \\ n-l \text{ odd}}}^{n} \binom{n}{l} \\
&= \frac{n}{2}(2^n - 1) - \frac{1}{2} \sum_{l=1}^{n} l \binom{n}{l} - \frac{1}{2} \sum_{\substack{l=1 \\ n-l \text{ odd}}}^{n} \left[ \binom{n-1}{l-1} + \binom{n-1}{l} \right] \\
&= \frac{n}{2}(2^n - 1) - n2^{n-2} - 2^{n-2} + \left( \frac{1}{4} - (-1)^n \frac{1}{4} \right) \\
&= (n-1)2^{n-2} - \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

We turn to (4.6.6) and prove first a combinatorial formula for a lacunary sum of binomial coefficients, see e.g. [Rio68, Section 4, Problem 8, p.161]. Let $\xi$ be a primitive third root of unity in $\mathbb{C}$, then the following holds

$$\begin{aligned}
3 \sum_{l \equiv r \ (3)} \binom{n}{l} &= \sum_{l=0}^{n} \binom{n}{l} (1 + \xi^{l-r} + \xi^{2l-2r}) \\
&= 2^n + \xi^{-r}(1+\xi)^n + \xi^{-2r}(1+\xi^2)^n \\
&= 2^n + (-1)^n (\xi^{n+r})^2 + (-1)^n \xi^{n+r}.
\end{aligned}$$

Write $n = 3a + b$ for integers $a \in \mathbb{Z}_{\geq 0}$ and $b \in \{0, 1, 2\}$. Then we get

$$\begin{aligned}
\sum_{l=1}^{n} \binom{n}{l} \left\lfloor \frac{n-l}{3} \right\rfloor &= \sum_{l=1}^{n} \frac{n-l}{3} \binom{n}{l} - \frac{1}{3} \sum_{\substack{n-l \equiv 1 \ (3) \\ l \neq 0}} \binom{n}{l} - \frac{2}{3} \sum_{\substack{n-l \equiv 2 \ (3) \\ l \neq 0}} \binom{n}{l} \\
&= \frac{n}{3}(2^{n-1} - 1) - \frac{1}{3} \sum_{l \equiv 1 \ (3)} \binom{n}{l} - \frac{2}{3} \sum_{l \equiv 2 \ (3)} \binom{n}{l} + \frac{b}{3} \\
&= \frac{n-2}{3} 2^{n-1} - \frac{n}{3} + \frac{b}{3} - \frac{(-1)^n}{9} \left( \xi^{2b+2} + \xi^{b+1} + 2\xi^{2b+1} + 2\xi^{b+2} \right).
\end{aligned}$$

A simple computation shows that

$$\frac{b}{3} - \frac{(-1)^n}{9} \left( \xi^{2b+2} + \xi^{b+1} + 2\xi^{2b+1} + 2\xi^{b+2} \right) = \delta,$$

which proves (4.6.6) and thus the lemma. □

*Proof of Theorem 4.1.1.* The statement for $d = 4$ follows from [Tot16a] for $N \leq 4$ (see also [Sch19b, Theorem 1.1]) and [PS23, Theorem 1.1] for $N = 5$, see also

Example 4.3.6.

Let now $d \geq 5$ and $N \leq \frac{d+1}{16}2^d$ be positive integers. If $3 \leq N \leq (d-2)+2^{d-2}-2$, then the theorem follows from [Sch19b, Theorem 1.1], because we can then write $N$ uniquely as $N = n + r$ for integers $n, r \geq 1$ satisfying $2 \leq n \leq d-2$ and $2^{n-1}-2 \leq r \leq 2^n - 2$. Hence we can assume that $N \geq d-4+2^{d-2}$. Let $n = d-2$, $r = 2^{d-2}-2$ and $s = N - n - r$ be non-negative integers. We claim that

$$s \leq (d-3)2^{d-4} - \left\lfloor \frac{d-2}{2} \right\rfloor .$$

Indeed, otherwise we get

$$N = n + r + s > d - 4 + 2^{d-2} + (d-3)2^{d-4} - \left\lfloor \frac{d-2}{2} \right\rfloor \geq \frac{d+1}{16}2^d,$$

which yields a contradiction to the assumption $N \leq \frac{d+1}{16}2^d$. Thus, Theorem 4.6.1 implies by Lemma 4.6.4 that $\mathrm{Tor}(X_d)$ is divisible by 2 for a very general degree $d$ hypersurface $X_d \subset \mathbb{P}^{N+1}$. In particular, $X_d$ does not admit an integral decomposition of the diagonal. $\square$

*Proof of Theorem 4.1.2.* Let $d \geq 5$ and $3 \leq N \leq \frac{d+1}{48}2^d$ be integers. If $3 \leq N \leq (d-3)+2^{d-3}-2$, then the theorem follows from [Sch21a, Theorem 7.1], because we can write $N = n + r$ for unique positive integers $n, r$ satisfying $2 \leq n \leq d-3$ and $2^{n-1}-2 \leq r \leq 2^n - 2$. Hence we can assume that $N \geq d-5+2^{d-3}$. Let $n = d-3$, $r = 2^{d-3}-2$ and $s = N - n - r$ be non-negative integers. We claim that

$$s \leq \frac{d-5}{3}2^{d-4} - \frac{d-3}{3} + \delta,$$

where $\delta$ is defined as in Lemma 4.6.4. (Note that $n = d-3$.) Indeed otherwise

$$N = n+r+s > d-3+2^{d-3}-2+\frac{d-5}{3}2^{d-4}-\frac{d-3}{3}+\delta = \frac{d+1}{48}2^d+\frac{2d}{3}+\delta-4 \geq \left\lfloor \frac{d+1}{48}2^d \right\rfloor,$$

which yields a contradiction to the assumption that $N$ is an integer satisying the inequality $N \leq \frac{d+1}{48}2^d$. Thus, Theorem 4.6.1 implies together with Lemma 4.6.4 that $3 \mid \mathrm{Tor}(X_d)$ for a very general degree $d$ hypersurface $X_d \subset \mathbb{P}^{N+1}$. In particular, $X_d$ has no an integral decomposition of the diagonal. $\square$

*Proof of Theorem 4.1.3.* This is Theorem 4.6.1 with the bound in Lemma 4.6.3. $\square$

# 5. Torsion order of complete intersections

## 5.1. Introduction

The torsion order $\mathrm{Tor}(X)$ of an integral variety $X$ over a field $k$ is the smallest positive integer $e \in \mathbb{Z}_{\geq 1}$ such that a decomposition

$$e \cdot \Delta_X = z \times X + Z \in \mathrm{CH}_{\dim X}(X \times_k X) \tag{5.1.1}$$

exists, where $z \in \mathrm{CH}_0(X)$ is a zero-cycle and $Z$ is a cycle on $X \times_k X$ whose support does not dominate the second factor; we set $\mathrm{Tor}(X) = \infty$ if no such decomposition exists. The notion originates from work of Bloch [Blo80] (using an idea of Colliot-Thélène) and Bloch–Srinivas [BS83] and has been studied for instance in [CL17, Kah17, Sch21a, LS25]. Slight variants of the torsion order have been considered in [ACTP17] and [Voi17].

A rationally (chain) connected smooth projective variety $X$ has finite torsion order and $X$ admits a decomposition of the diagonal (meaning $\mathrm{Tor}(X) = 1$) if $X$ is (retract) rational or $\mathbb{A}^1$-connected. Thus the non-rationality of smooth projective (rationally connected) varieties can be shown by proving that the torsion order is strictly larger than 1 (or equivalently disproving a decomposition of the diagonal). This approach to the rationality problem has been successfully started by Voisin [Voi15]. The rationality problem itself has been extensively studied for a very general member of many classes of algebraic varieties using various methods. We refer the reader to the recent surveys [Deb24, Sch25b] for an overview of the methods and results.

Two other properties of a smooth projective integral variety are captured by its torsion order. If $f\colon X \longrightarrow Y$ is a generically finite morphism between smooth projective integral varieties over a field $k$, then a "pull-push-argument" shows that $\mathrm{Tor}(Y)$ divides $\deg f \cdot \mathrm{Tor}(X)$. In particular, $\mathrm{Tor}(Y)$ provides a lower bound for the degree of any unirational parametrization of $Y$. The torsion order of a smooth projective integral $k$-variety is also the smallest positive integer such that the kernel of the degree morphism $\deg\colon \mathrm{CH}_0(X_L) \longrightarrow \mathbb{Z}$ is $e$-torsion for any field extension $L/k$, where $\mathrm{Tor}(X) = \infty$ if and only if no such integer exists.

In this chapter, we provide new and improved lower bounds for the torsion order of very general complete intersections in projective space as well as hypersurfaces in products of projective spaces and Grassmannians.

### 5.1.1. Complete intersections

An argument of Rojtman [Roi80] for hypersurfaces extended to complete intersections by Chatzistamatiou–Levine [CL17, Proposition 4.1] shows that the torsion order of a Fano complete intersection of multidegree $(d_1, \dots, d_s)$ in $\mathbb{P}^{N+s}$ over any field divides $d_1! \cdot \dots \cdot d_s!$, in particular it is finite.

In the same paper, Chatzistamatiou–Levine provide a lower bound for very general Fano complete intersections by building on earlier work of Totaro [Tot16a] and Kollár [Kol95]. The torsion order of a very general complex complete intersection $X$ of multidegree $(d_1, \dots, d_s)$ and dimension $N$ is divisible by some prime power $p^{m'}$ if

$$d_i \geq p^{m'} \left\lceil \frac{N + s + 1 - d_1 - \dots - d_s + d_i}{p^{m'} + 1} \right\rceil$$

for some $i \in \{1, \dots, s\}$ if $p$ is odd or $N$ is even, see [CL17, Theorem 7.2]. By using the Fano index $r(X) \coloneqq \dim X + s + 1 - d_1 - \dots - d_s$, the statement roughly means that $\mathrm{Tor}(X)$ is divisible by a positive integer $m$ if some degree $d_i$ satisfies

$$d_i \geq m \cdot r(X) + m.$$

For hypersurfaces, this bound was significantly improved by Schreieder [Sch19b, Sch21a]: A very general hypersurface $X$ of degree $d$ and Fano index $r(X) > 0$ over a field $k$ has torsion order divisible by a integer $m > 0$, if $m$ is invertible in $k$ and

$$d \geq \log_2(r(X) + m) + m.$$

We extend the bound of Schreieder to complete intersections.

**Theorem 5.1.1.** *The torsion order of a very general complete intersection of multidegree $(d_1, \dots, d_s)$ and positive Fano index $r$ over a field $k$ is divisible by a positive integer $m$ if $m \in k^*$ and there exists a degree $d_i$ such that $d_i \geq \log_2(r + m) + m$.*

A complete intersection of multidegree $(d_1, \dots, d_s)$ is *very general* if it is (up to a field extension) isomorphic to the geometric generic fibre of the parameter space of complete intersection of the given multidegree, see Section 2.4.

In fact we obtain slightly better bounds, which recover the bounds for the torsion order of hypersurfaces in Chapter 4. For instance, we obtain the following result for the 2-divisibility of the torsion order, which yields the same bound in positive characteristic as the stable irrationality bound of Nicaise–Ottem [NO22] in characteristic 0. The results in [NO22] are proven using a motivic method, which builds on earlier work of Nicaise–Shinder [NS19] and Kontsevich–Tschinkel [KT19]. The motivic method was recently improved by Hotchkiss–Stapleton [HS25] to handle $\mathbb{A}^1$-connectedness.

**Theorem 5.1.2.** *Let $k$ be a field of characteristic different from $2$. Then the torsion order of a very general complete intersection of multidegree $(d_1, \dots, d_s)$, dimension at least $4$ and Fano index $r$ is divisible by $2$ if $r \leq 2$ or some degree $d_i \geq 4$ and*

$$r \leq (d_i + 1)2^{d_i - 4} - \left\lfloor \frac{d_i + 2}{2} \right\rfloor .$$

*In particular, such complete intersections are neither (retract) rational nor $\mathbb{A}^1$-connected.*

Note that the torsion order of complete intersections with non-positive Fano index $r$ might be infinite, e.g. non Fano complete intersection over $\mathbb{C}$ have infinite torsion order. The case of hypersurfaces in the above theorem is covered by [Moe23], who showed the stable irrationality in char 0, and Theorem 4.1.1. The latter results also cover the case of quartic fivefolds, see also [NO22] and [PS23, Theorem 1.1].

### 5.1.2. Hypersurfaces in rational varieties

We illustrate the flexibility of our approach by considering also hypersurfaces in other rational varieties. Among them hypersurfaces in products of projective spaces are probably the most extensively studied ones with regard to the rationality problem, see for example [BB18, AO18, ABBP20, NO22, Moe23].

To the best of our knowledge, there seems to be no upper bound for the torsion order of hypersurfaces in products of projective spaces known in the literature. Although one might be able to adapt Rojtman's argument for complete intersection also to these examples, which we do not pursue here. Instead we provide the following lower bound.

**Theorem 5.1.3.** *Let $k$ be a field and let $m, s \in \mathbb{Z}_{\geq 1}$ be positive integers such that $m$ is invertible in $k$. Then the torsion order of a very general hypersurface of mul-*

*tidegree* $(d_1, d_2, \dots, d_s)$ *in* $\mathbb{P}_k^{M_1} \times_k \cdots \times_k \mathbb{P}_k^{M_s}$ *is divisible by* $m$*, if*

$$M_1 \geq 4 \quad \text{and} \quad (d_1, d_2, \dots, d_s) \geq (\log_2(M_1) + m, M_2 + 1, M_3 + 1, \dots, M_s + 1).$$

*In particular, it is neither stably rational nor retract rational nor* $\mathbb{A}^1$*-connected.*

This extends and generalizes earlier results by Nicaise–Ottem [NO22, Proposition 6.2], where stable irrationality for $s = 2$ is shown in characteristic 0.

A different class of examples are hypersurfaces in Grassmannians, which are related to Gushel-Mukai varieties, see e.g. [DK18]. We obtain the following result by relating certain hypersurfaces in Grassmannians directly with the ones in [Sch19b, LS25].

**Theorem 5.1.4.** *Let* $l, n \in \mathbb{Z}_{\geq 1}$ *be positive integers such that* $l(n-l) \geq 4$*. Consider the complex Grassmannian* $\mathrm{Gr}(l, n)$ *and fix a Plücker embedding* $\mathrm{Gr}(l, n) \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$*. Then the torsion order of the intersection of* $\mathrm{Gr}(l, n)$ *with a very general complex hypersurface of degree* $d \geq 4$ *in* $\mathbb{P}^N_{\mathbb{C}}$ *is divisible by* 2*, if* $l(n-l) \leq (d+1)2^{d-4}$*.*

The restriction to the complex numbers is mostly for simplicity of the argument and it should be possible to obtain a similar result also in positive characteristic, see Remark 5.4.10.

The stable rationality for hypersurfaces in the Grassmannian $\mathrm{Gr}(2, n)$ is studied in an unpublished work of Ottem[1] and also in work of Yoshino [Yos24, Yos25]. For $l = 2$, Yoshino obtained a slightly better bound $(2(n-2) \leq (d+1)2^{d-4} + 2)$ for the stable irrationality of hypersurfaces in $\mathrm{Gr}(2, n)$, see [Yos25, Theorem 1.3].

### 5.1.3. Outline of the argument

Nicaise–Ottem [NO22] proves the (stable) irrationality of algebraic varieties using a degeneration to snc schemes, where the obstruction to rationality lies in some lower stratum using the motivic volume. A cycle-theoretic analogue of this method was given first by Pavic–Schreieder [PS23] for proper families with a generalization to non-proper families by Lange–Schreieder [LS25], see Chapter 4.

We elaborate the idea of [LS25] to consider non-proper degenerations to snc schemes by studying affine degenerations to which the method of [PS23, LS25] is applicable. In fact, we reformulate most of the assumptions in terms of basic commutative algebra statements. An iterated application of this affine degeneration

[1] available at his webpage: https://www.mn.uio.no/math/personer/vit/johnco/papers/gr25.pdf.

allows us to construct affine complete intersections from affine hypersurfaces, where the torsion order of the former is controlled by the torsion order of the latter.

Applying this machinery to the concrete hypersurface examples from [Sch19b, Sch21a, LS25] provides affine complete intersections with a (non-trivial) lower bound on the torsion order. We emphasize that this ultimately relies on certain vanishing results for unramified cohomology classes found for instance in [Sch19a, Sch19b].

The main results are obtained by considering different compactifications of these affine examples together with a standard degeneration argument. To control the compactification, we use the theory of Gröbner basis.

We would like to point out that we use degenerations similar to and inspired by the ones in [NO22, Moe23]. While they use one degeneration into many components, which they control via tropical geometry, we consider several simple degenerations into 2 components.

The framework of affine degenerations is developed in Section 5.2. In Section 5.3, we check that the hypersurfaces from [Sch19b, Sch21a, LS25] as well as special complete intersections related to an example of Hassett–Pirutka–Tschinkel [HPT18a] fit in our framework of affine degeneration. This essentially boils down to some elementary computations using the explicit equations. The final section (Section 5.4) is devoted to the proof of the main results.

## 5.2. Affine degenerations

In this section, we construct (strictly semi-stable) families over a dvr, which satisfy the assumptions of Theorem 4.3.8. In particular, we need to write down a family whose special fibre consists of two irreducible components and whose total space is regular. This is straightforward in the affine setting. Let $A$ be a smooth $k$-algebra of finite type and let $f_1, f_2 \in A$, then the natural morphism

$$\operatorname{Spec}\left(A \otimes_k k[t]_{(t)}/(t - f_1 f_2)\right) \longrightarrow \operatorname{Spec} k[t]_{(t)} \tag{5.2.1}$$

is a family over the dvr $k[t]_{(t)}$ with regular total space such that the special fibre is equal to

$$\operatorname{Spec} A/(f_1 f_2) \simeq \operatorname{Spec}\left(A/(f_1)\right) \cup \operatorname{Spec}\left(A/(f_2)\right).$$

A key assumption in Theorem 4.3.8 is the "rationality" condition (3), which we reformulate to an algebraic condition for degenerations as (5.2.1) in the first part. In the second part, we provide two general constructions of families as in (5.2.1) which satisfy the assumptions in Theorem 4.3.8.

### 5.2.1. Strongly rational algebras

The assumption (3) in Theorem 4.3.8 states that the irreducible components of the special fibre are isomorphic to an open subscheme of affine space. Rephrasing this geometric condition algebraically leads to the following definition.

**Definition 5.2.1.** Let $k$ be a field. A finite type $k$-algebra $B$ is called *strongly $k$-rational* if $B$ is isomorphic as a $k$-algebra to the localization of a polynomial ring over $k$ and $B \neq 0$.

**Example 5.2.2.** (1) The polynomial ring $k[x_1, \dots, x_n]$ is clearly a strongly $k$-rational algebra for every $n \geq 0$.

(2) The finite type $k$-algebra $k[x_1, x_2]/(1 + x_1x_2)$ is strongly $k$-rational, as

$$k[x_1, x_2]/(1 + x_1x_2) \simeq k[x_2, x_2^{-1}].$$

(3) If $B$ is a strongly $k$-rational algebra, then $B \otimes_k L$ is strongly $L$-rational for every field extension $L/k$.

The following remark verifies that the spectrum of a strongly rational $k$-algebra is isomorphic to an open in affine space, in particular rational.

**Remark 5.2.3.** Let $B$ be a strongly $k$-rational algebra. Then there exists a polynomial ring $A = k[x_1, \dots, x_n]$ and a multiplicative subset $S \subset A$ such that $B \simeq S^{-1}A$. Since $B$ is also of finite type over $k$ by the definition of strongly rational $k$-algebra, we find that $\operatorname{Spec} B \to \operatorname{Spec} A = \mathbb{A}^n_k$ is an open immersion by [Oda04, Theorem 2.7], see also the Added part in *loc. cit.* for an alternative argument[2].

The above remark implies that strongly $k$-rational algebras are geometrically integral and smooth $k$-algebras. We provide an algebraic proof for the convenience of the reader.

**Lemma 5.2.4.** *A strongly $k$-rational algebra $B$ is a geometrically integral, smooth $k$-algebra.*

*Proof.* It suffices to prove that $B$ is an integral domain and (formally) smooth over $k$, which is obvious as $B$ is by definition isomorphic as a $k$-algebra to the localization of a polynomial ring over $k$. ☐

[2] We found the argument and reference through an answer of Martin Brandenburg on mathoverflow.

### 5.2.2. Affine strictly semi-stable families

We provide a general setup to construct affine strictly semi-stable schemes over the dvr $k[t]_{(t)}$ such that its special fibre consists of two irreducible components which are rational.

**Definition 5.2.5.** Let $k$ be a field and let $A$ be an integral smooth $k$-algebra of finite type and of Krull dimension $N+1$. We say that two elements $f_1 \in A[z] := A \otimes_k k[z]$ and $f_2 \in A$ are *admissible with respect to* $A$ if the quotient ring $A[z]/(f_1, f_2)$ is a geometrically integral $k$-algebra of dimension $N$ and the $k$-algebras $A[z]/(f_1)$ and $A/(f_2)$ are strongly $k$-rational and of Krull dimension $N+1$ and $N$, respectively.

The above definition is motivated by the following proposition.

**Proposition 5.2.6.** *Let $k$ be a field and let $A$ be an integral smooth $k$-algebra of finite type and of Krull dimension $N+1$. Assume that $f_1 \in A[z]$ and $f_2 \in A$ are admissible with respect to $A$ in the sense of Definition 5.2.5. Then the affine scheme*

$$\operatorname{Spec}(A[z]_{\partial_z f_1} \otimes_k R)/(t - f_1 f_2) \tag{5.2.2}$$

*is a strictly semi-stable $R := k[t]_{(t)}$-scheme. Moreover, the $k(t)$-algebra*

$$A[z] \otimes_k k(t)/(t - f_1 f_2) \tag{5.2.3}$$

*is an integral domain.*

*Proof.* We check the definition of strictly semi-stable $R$-schemes, see Definition 2.3.2. Note first that the $k[t]$-algebra

$$B := A[z,t]/(t - f_1 f_2) \simeq A[z]$$

is a smooth $k$-algebra and thus in particular flat. As $k[t]$ is an unramified $k$-algebra, $B$ is a flat $k[t]$-algebra by [GW23, Remark 18.30]. Thus the $R$-algebra

$$B \otimes_{k[t]} R = (A[z] \otimes_k R)/(t - f_1 f_2)$$

is flat. Localizing at the multiplicative subset $\{(\partial_z f_1)^n : n \in \mathbb{N}\} \subset B \otimes_{k[t]} R$, then implies that (5.2.2) is a flat $R$-scheme. Since $B$ is clearly an integral domain and being an integral domain is preserved under localization, it follows that the $k(t)$-algebra (5.2.3) is an integral domain and the affine scheme (5.2.2) is integral.

The generic fibre of the $R$-scheme (5.2.2) is the spectrum of the $k(t)$-algebra

$$(A \otimes_k k(t))[z,w]/(t - f_1 f_2, w\partial_z f_1 - 1), \tag{5.2.4}$$

which is a standard smooth $A \otimes_k k(t)$-algebra in the sense of [Stacks, Tag 00T6]. Thus (5.2.4) is a smooth $k(t)$-algebra, as $A$ is a smooth $k$-algebra by assumption.

The special fibre of the $R$-scheme (5.2.2) has two irreducible components, namely $\operatorname{Spec} A[z]_{\partial_z f_1}/(f_1)$ and $\operatorname{Spec} A[z]_{\partial_z f_1}/(f_2)$, which are both smooth $k$-schemes by Lemma 5.2.4 and clearly Cartier divisors in (5.2.2). The intersection is the spectrum of the $k$-algebra

$$A[z]_{\partial_z f_1}/(f_1, f_2) \simeq (A/(f_2))\,[z,w]/(f_1, w\partial_z f_1 - 1),$$

which is a smooth $k$-algebra by a similar argument as for (5.2.4). Note that this uses the fact that $A/(f_2)$ is strongly $k$-rational and thus a smooth $k$-algebra by Lemma 5.2.4. □

The following example provides a way to construct admissible pairs with respect to strongly $k$-rational algebras.

**Example 5.2.7.** Let $k$ be a field and let $B$ be a strongly $k$-rational algebra, e.g. a polynomial ring. Let $f \in B[z]$ such that $B[z]/(f)$ is geometrically integral of Krull dimension $\dim B$. Then the pair $f_1 = f + y \in B[y,z]$ and $f_2 = y \in B[y]$ is admissible with respect to $B[y]$. Indeed, the assumptions on $B$ and $f$ together with the $k$-algebra isomorphisms

$$B[y,z]/(f_1,f_2) \simeq B[z]/(f), \quad B[y,z]/(f_1) = B[y,z]/(f+y) \simeq B[z], \quad B[y]/(f_2) = B$$

imply directly that $f_1$ and $f_2$ are admissible with respect to $B[y]$.

### 5.2.3. The key construction

In this part, we provide the key construction of a new admissible pair from an admissible pair. The idea is the following simple observation: Let $f_1 \in A[z]$ and $f_2 \in A$ be admissible with respect to a strongly $k$-rational algebra $A$ for some field $k$. Then the generic fibre of the family (5.2.1) is the spectrum of the $k(t)$-algebra

$$A[z] \otimes_k k(t)/(t - f_1 f_2) \simeq A[z,y] \otimes_k k(t)/(t + f_1 y, f_2 + y). \tag{5.2.5}$$

Note that the $k(t)$-algebras

$$A[z,y]\otimes_k k(t)/(t+f_1y) \simeq (A[z]\otimes_k k(t))_{f_1} \quad \text{and} \quad A[z,y]\otimes_k k(t)/(f_2+y) \simeq A[z]\otimes_k k(t)$$

are strongly $k$-rational. Thus $g_1 \coloneqq t + f_1 y$ and $g_2 \coloneqq f_2 + y$ are admissible with respect to $A[z,y] \otimes_k k(t)$ if the $k(t)$-algebra (5.2.5) is geometrically integral.

As integral varieties over a field $K$ with a $K$-rational point in the smooth locus are geometrically integral, see e.g. [Stacks, Tag 0CDW], we introduce the following extra condition for $f_1 \in A[z]$ and $f_2 \in A$: for every field extension $F/k$ and every $(q_1, q_2) \in F^2$ there exists an F-algebra epimorphism

$$A[z]_{\partial_z f_1} \otimes_k F/(f_1 + q_1, f_2 + q_2) \twoheadrightarrow F. \tag{$\star$}$$

The following lemma shows that this extra condition guarantees that the $k(t)$-algebra (5.2.5) admits a $k(t)$-rational point in the smooth locus.

**Lemma 5.2.8.** *Let $k$ be a field and let $A$ be an integral smooth $k$-algebra of finite type. Assume that $f_1 \in A[z]$ and $f_2 \in A$ are admissible with respect to $A$ in the sense of Definition 5.2.5 and satisfy condition ($\star$). Then the $k(t)$-scheme*

$$\mathrm{Spec}(A[z]_{\partial_z f_1} \otimes_k k(t))/(t - f_1 f_2)$$

*contains a $k(t)$-rational point and is geometrically integral over $k(t)$.*

*Proof.* The existence of a $k(t)$-rational point is equivalent to showing that the $k(t)$-algebra

$$(A[z]_{\partial_z f_1} \otimes_k k(t))/(t + f_1 y, f_2 + y)$$

admits a $k(t)$-algebra epimorphism to $k(t)$. The latter can be constructed as

$$(A[z,y]_{\partial_z f_1} \otimes_k k(t))/(t + f_1 y, f_2 + y) \twoheadrightarrow (A[z]_{\partial_z f_1} \otimes_k k(t))/(t + f_1, f_2 + 1) \twoheadrightarrow k(t),$$

where the first surjection is the quotient by the ideal $(y-1)$ and the second surjection exists by ($\star$) applied to the field extension $F = k(t)$ and the pair $(q_1, q_2) = (t, 1)$.

Thus the smooth and integral $k(t)$-variety

$$\mathrm{Spec}(A[z]_{\partial_z f_1} \otimes_k k(t))/(t - f_1 f_2),$$

see Proposition 5.2.6, is geometrically integral over $k(t)$ by [Stacks, Tag 0CDW]. $\square$

**Corollary 5.2.9.** *Let $k$ be a field and let $A$ be a strongly $k$-rational algebra. If $f_1 \in A[z]$ and $f_2 \in A$ are admissible with respect to $A$ in the sense of Definition 5.2.5 and satisfy the condition ($\star$), then the two pairs $(h_1, h_2)$ and $(\tilde{h}_1, \tilde{h}_2)$ with*

$$\begin{aligned} h_1 &\coloneqq t + f_1 y \in A[y,z] \otimes_k k(t) \quad \textit{and} \quad h_2 \coloneqq f_2 + y \in A[y] \otimes_k k(t) \\ \tilde{h}_1 &\coloneqq f_1 + y \in A[y,z] \otimes_k k(t) \quad \textit{and} \quad \tilde{h}_2 \coloneqq t + f_2 y \in A[y] \otimes_k k(t) \end{aligned}$$

*are admissible with respect to $A[y] \otimes_k k(t)$ and also satisfy ($\star$).*

*Proof.* We check the definition of an admissible pair: Note that

$$A[y,z] \otimes_k k(t)/(h_1, h_2) \simeq A[z] \otimes_k k(t)/(t - f_1 f_2) \simeq A[y,z] \otimes_k k(t)/(\tilde{h}_1, \tilde{h}_2)$$

Thus the quotient is geometrically integral over $k(t)$ by Lemma 5.2.8. Recall that for $i = 1, 2$ the $k(t)$-algebras

$$A[z,y] \otimes_k k(t)/(t + f_i y) \simeq (A[z] \otimes_k k(t))_{f_i} \quad \text{and} \quad A[z,y] \otimes_k k(t)/(f_i + y) \simeq A[z] \otimes_k k(t)$$

are strongly $k(t)$-rational. Thus the pairs $(h_1, h_2)$ and $(\tilde{h}_1, \tilde{h}_2)$ are admissible with respect to $A[y] \otimes_k k(t)$. To show condition ($\star$), let $F/k(t)$ be a field extension and let $(q_1, q_2) \in F^2$. Then the composition

$$A[y,z]_{\partial_z h_1} \otimes_k F/(h_1 + q_1, h_2 + q_2) \longrightarrow A[z]_{\partial_z f_1} \otimes_k F/(t + f_1 + q_1, f_2 + 1 + q_2) \xrightarrow{(\star)} F$$

yields an $F$-algebra epimorphism, where the first map is the quotient by the ideal $(y-1)$ and the second map exists because $f_1$ and $f_2$ satisfy the condition ($\star$). Hence, the pair $(h_1, h_2)$ satisfies condition ($\star$). The same argument works for $(\tilde{h}_1, \tilde{h}_2)$. $\square$

We end this part by reformulating ($\star$) for the construction in Example 5.2.7.

**Example 5.2.10.** Let $k$ be a field and let $B$ be a strongly $k$-rational algebra, e.g. a polynomial ring. Let $f \in B[z]$ be neither a zero-divisor nor a unit. Assume that $B[z]/(f)$ is geometrically integral and that $f$ satisfies the condition

$$\forall F/k \; \forall q \in F \; \exists \text{ F-algebra epimorphism } B[z]_{\partial_z f} \otimes_k F/(f + q) \twoheadrightarrow F. \qquad (\star\star)$$

Then the pair $f_1 = f + y \in B[y,z]$ and $f_2 = y \in B[y]$ is admissible with respect to $B[y]$ by Example 5.2.7 and satisfies condition ($\star$), as

$$B[y,z]_{\partial_z (f+y)} \otimes_k F/(f + y + q_1, y + q_2) \simeq B[z]_{\partial_z f} \otimes_k F/(f - q_2 + q_1)$$

for every field extension $F/k$ and $q_1, q_2 \in F$.

### 5.2.4. Admissible pairs and torsion orders

In this part, we apply Theorem 4.3.8 to the degeneration (5.2.2), which provides us a flexible way to understand the torsion order of complete intersections by understanding the torsion order of hypersurfaces.

The first theorem is a key technical result, which controls the torsion order in affine degeneration of the form (5.2.2) and can be seen as an algebraic version of Theorem 4.3.8.

**Theorem 5.2.11.** *Let $k$ be an algebraically closed field and let $\Lambda$ be a commutative ring with $1$ and of finite characteristic $e \in \mathbb{Z}_{\geq 1}$ such that $e \in k^*$. Let $A$ be an integral smooth $k$-algebra of finite type. Assume that $f_1 \in A[z]$ and $f_2 \in A$ are admissible with respect to $A$ and satisfy condition ($\star$). Let $l \in A[z]$ be a non-trivial element and consider*

$$W_Z := \operatorname{Spec} A[z]/(f_1, f_2, l \cdot \partial_z f_1) \subset Z := \operatorname{Spec} A[z]/(f_1, f_2)$$
$$W_{\bar{X}} := \operatorname{Spec} A[z] \otimes_k \overline{k(t)}/(t - f_1 f_2, l \cdot \partial_z f_1) \subset \bar{X} := \operatorname{Spec} A[z] \otimes_k \overline{k(t)}/(t - f_1 f_2),$$

*where $\overline{k(t)}$ is an algebraic closure of a purely transcendental extension $k(t)/k$. Then $\bar{X}$ is an integral variety and $\operatorname{Tor}^{\Lambda}(Z, W_Z) \mid \operatorname{Tor}^{\Lambda}(\bar{X}, W_{\bar{X}})$.*

*Proof.* Let $R := k[t]_{(t)}$ and consider the $R$-scheme

$$\mathcal{X} := \operatorname{Spec}\left(A[z] \otimes_k R/(t - f_1 f_2)\right). \tag{5.2.6}$$

Note that $Z$ is the intersection of the two irreducible components of the special fibre and $\bar{X}$ is the geometric generic fibre of $\mathcal{X}$. We aim to apply Theorem 4.3.8 to the family (5.2.6); thus we check that the assumptions in Theorem 4.3.8 are satisfied.

The family (5.2.6) is flat and separated over $R$ by [GW20, Proposition 14.20 and Proposition 9.15]. It is clearly of finite type.

The generic fibre $X$ is geometrically integral, as it is integral by Proposition 5.2.6, Cohen-Macaulay by [Eis95, Proposition 18.13], and contains an open (dense) subscheme, which is geometrically integral by Lemma 5.2.8.

Consider the closed affine subscheme

$$W_{\mathcal{X}} := \operatorname{Spec}\left(A[z] \otimes_k R/(t - f_1 f_2, l \cdot \partial_z f_1)\right) \subset \mathcal{X}$$

Note that $W_{\mathcal{X}} \times_{\mathcal{X}} Z = W_Z$ and $W_{\mathcal{X}} \times_{\mathcal{X}} \bar{X} = W_{\bar{X}}$ for $W_Z$ and $W_{\bar{X}}$ as in the statement of the theorem. The open subscheme $\mathcal{X}^\circ := \mathcal{X} \setminus W_{\mathcal{X}} \subset \mathcal{X}$ is given by

$$\operatorname{Spec}\left(A[z] \otimes_k R/(t - f_1 f_2)\right)_{l \cdot \partial_z f_1} = \operatorname{Spec} A[z]_{l \cdot \partial_z f_1} \otimes_k R/(t - f_1 f_2),$$

where we used that $l, f_1 \in A[z]$. Thus $\mathcal{X}^\circ$ is a strictly semi-stable $R$-scheme by Proposition 5.2.6 and so condition (2) of Theorem 4.3.8 is satisfied. The special fibre of the family (5.2.6) consists of the two irreducible varieties

$$Y_0 \simeq \operatorname{Spec} A[z]/(f_1) \quad \text{and} \quad Y_1 \simeq A[z]/(f_2).$$

By definition of an admissible pair, the intersection $Z = Y_0 \cap Y_1 \simeq \operatorname{Spec} A[z]/(f_1, f_2)$ is (geometrically) integral and the varieties $Y_0$ and $Y_1$ are isomorphic to an open subscheme of affine space, see Remark 5.2.3. Hence condition (1) and (3) hold. In total, we have shown that the assumptions of Theorem 4.3.8 are satisfied for the family $\mathcal{X}$ and the closed subscheme $W_{\mathcal{X}}$. Thus the theorem follows from Theorem 4.3.8. □

The following theorem provides a framework to inductively construct certain affine complete intersection from affine hypersurfaces by simultaneously controlling the (relative) torsion order. As the torsion order is related to the rationality, see the discussion in the introduction, the proposition can be seen as an affine analogue and generalization of [NO22, Theorem 7.7], where stable irrationality is studied.

**Theorem 5.2.12.** *Let $k$ be an uncountable algebraically closed field and let $m \geq 2$ and $n, r \geq 0$ be integers with $m \in k^*$. If there exist non-constant polynomials*

$$f_1, \dots, f_r \in k[x_1, \dots, x_{n+r}] \quad \text{and} \quad f \in k[x_1, \dots, x_{n+r}, z]$$

*such that*

*(C1) the $k$-algebra $B := k[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r)$ is strongly $k$-rational;*

*(C2) the affine scheme $\operatorname{Spec} B[z]/(f)$ is an integral $k$-variety of dimension $\dim B$;*

*(C3) there exists a non-zero polynomial $l \in k[x_1, \dots, x_{n+r}, z]$ such that*

$$\operatorname{Tor}^{\mathbb{Z}/m}\left(\operatorname{Spec} B[z]/(f), \operatorname{Spec} B[z]/(f, l \cdot \partial_z f)\right) = m;$$

*(C4) $f \in B[z]$ satisfies condition ($\star\star$).*

*Then for every integers $d$ and $M$ with $d \geq M \geq 1$, there exist polynomials*

$$\check{f}, \tilde{f} \in k[x_1, \ldots, x_{n+r}, w_1, \ldots, w_M, z] \quad \textit{and} \quad f_{r+1} \in k[w_1, \ldots, w_M]$$

*of degree $\deg \check{f} = \deg f + d$, $\deg \tilde{f} = \deg f$ and $\deg f_{r+1} = d$ such that*

(a) *the polynomials $f_1, \ldots, f_{r+1}$ and $\tilde{f}$ satisfy the properties (C1)–(C4) if additionally $d = M$ or $M \geq 2$;*

(b) *the polynomials $f_1, \ldots, f_r$ and $\check{f}$ satisfy the properties (C1)–(C4) and the degrees with respect to the $x, z$- and $w$-coordinates are $\deg_{x,z} \check{f} = \deg f$ and $\deg_w \check{f} = d + 1$.*

We call $d$ the *added degree* and $M$ the *number of added variables.*

*Proof.* The theorem is a consequence of the following claim, which we prove by applying inductively the results from Section 5.2 together with a simple degeneration argument.

**Claim.** For all integers $d$ and $M$ with $d = M \geq 1$ or $d > M \geq 2$, there exist polynomials

$$\tilde{f} \in k[x_1, \ldots, x_{n+r}, z, w_1, \ldots, w_M] \quad \text{and} \quad f_{r+1} \in k[w_1, \ldots, w_M]$$

of degree $\deg f$ and $d$ respectively, such that $\tilde{f}$ and $f_{r+1}$ are admissible with respect to $B[w_1, \ldots, w_d]$ for $B$ as in (C1) and satisfy condition ($\star$) from Section 5.2 as well as

$$\mathrm{Tor}^{\mathbb{Z}/m}\Big(\operatorname{Spec} B[z, w_1, \ldots, w_M]/(\tilde{f}, f_{r+1}), \operatorname{Spec} B[z, w_1, \ldots, w_M]/(\tilde{f}, f_{r+1}, l \cdot \partial_z \tilde{f})\Big) = m$$

for some $l \in k[x_1, \ldots, x_{n+r}, z]$. Additionally, we have $\tilde{f} - f \in k[w_1, \ldots, w_M]$ of degree 1.

The claim immediately implies (a), see Definition 5.2.5 and ($\star$). The explicit construction of $\tilde{f}$ and $f_{r+1}$ in the claim allows us to construct $\check{f}$ and deduce (b).

The proof of the claim is divided into two steps. In the first step we show the claim for $d = M \geq 1$ via the results from Section 5.2. The second step proves the claim for $d > M \geq 2$ by applying a simple degeneration argument together with Lemma 4.2.9.

**Step 1.** Assume that $d = M \geq 1$. We construct inductively $\tilde{f}$ and $f_{r+1}$. If $d = 1$, then Example 5.2.10 applied to the strongly $k$-rational algebra $B$ and $f \in B[z]$ shows that

$$\tilde{f} = f + w \in k[x_1, \dots, x_{n+r}, w, z] \quad \text{and} \quad f_{r+1} = w \in k[w]$$

are admissible with respect to $B[w]$ and satisfy the condition ($\star$). Note that

$$B[w, z]/(\tilde{f}, f_{r+1}) = B[z]/(f),$$

which shows the claim for $d = 1$.

Suppose that for some $d \geq 1$ there exists polynomials $\tilde{f} \in k[x_1, \dots, x_{n+r}, w_1, \dots, w_d, z]$ and $f_{r+1} \in k[w_1, \dots, w_d]$ as in the claim with $M = d$. Consider a purely transcendental field extension $k(t)/k$. Then the polynomials

$$\begin{aligned} \tilde{g} &\coloneqq \tilde{f} + w_{d+1} \in k(t)[x_1, \dots, x_{n+r}, w_1, \dots, w_{d+1}, z] \quad \text{and} \\ g_{r+1} &\coloneqq t + w_{d+1} f_{r+1} \in k(t)[w_1, \dots, w_{d+1}] \end{aligned}$$

are admissible with respect to $B[w_1, \dots, w_{d+1}]$ and satisfy condition ($\star$) by Corollary 5.2.9. Note that

$$B' \coloneqq B \otimes_k k(t)[w_1, \dots, w_{d+1}, z]/(\tilde{g}, g_{r+1}) \simeq B \otimes_k k(t)[w_1, \dots, w_d, z]/(t - \tilde{f} f_{r+1}).$$

Thus, Theorem 5.2.11 and the assumption on the relative torsion order in the claim imply

$$\mathrm{Tor}^{\mathbb{Z}/m}\left(\operatorname{Spec} B' \otimes_{k(t)} \overline{k(t)}, \operatorname{Spec}(B' \otimes_{k(t)} \overline{k(t)})/(l \cdot \partial_z \tilde{g})\right) = m.$$

Up to identifying $\overline{k(t)}$ with $k$ via a field isomorphism, which exists as both fields are algebraically closed field extensions of the same (uncountable) transcendence degree over the same prime field, we can assume that the polynomials $\tilde{g}$ and $g_{r+1}$ are defined over $k$. Note that $\tilde{g}$ and $g_{r+1}$ are admissible with respect to $B[w_1, \dots, w_{d+1}]$ and satisfy condition ($\star$). We also have $\deg \tilde{g} = \deg \tilde{f}$, $\deg g_{r+1} = d + 1$, and $\deg_w \tilde{g} = 1$ as well as

$$\mathrm{Tor}^{\mathbb{Z}/m}(\operatorname{Spec} B[w_1, \dots, w_{d+1}, z]/(\tilde{g}, g_{r+1}), \operatorname{Spec} B[w_1, \dots, w_{d+1}, z]/(\tilde{g}, g_{r+1}, l \cdot \partial_z \tilde{g})) = m.$$

Thus, we have constructed inductively polynomials as in the claim for $d = M$, which concludes the first step.

**Step 2.** Assume that $2 \leq M < d$. By Step 1, there exist polynomials

$$\tilde{f} \in k[x_1, \dots, x_{n+r}, w_1, \dots, w_M, z] \quad \text{and} \quad f_{r+1} \in k[w_1, \dots, w_M]$$

of degree $\deg \tilde{f} = \deg f$ and $\deg f_{r+1} = M$, which satisfy the conditions in the claim for $d = M$. In fact, it follows from the construction in Step 1 that we can take

$$\begin{aligned} \tilde{f} &= f + w_1 + \cdots + w_M \in k[x_1, \dots, x_M, w_1, \dots, w_M, z] \text{ and} \\ f_{r+1} &= t_2 - w_M(t_1 - w_{M-1}g) \in k[w_1, \dots, w_M] \end{aligned} \tag{5.2.7}$$

for some $t_1 \in k$ with $t_1 = 0$ if and only if $M = 2$, some $t_2 \in k^*$ which is transcendental over the prime field of $k$, and some polynomial $g \in k[w_1, \dots, w_{M-2}]$ of degree $M-2$. Let $\overline{k(t)}$ be an algebraic closure of a purely transcendental field extension $k(t)/k$. Consider the polynomials

$$\begin{aligned} \tilde{g} &\coloneqq \tilde{f} \in \overline{k(t)}[x_1, \dots, x_N, w_1, \dots, w_M, z] \quad \text{and} \\ g_{r+1} &\coloneqq f_{r+1} + t w_M (w_{M-1} - \delta_{M,2}) h \in \overline{k(t)}[w_1, \dots, w_M], \end{aligned} \tag{5.2.8}$$

where the polynomials $\tilde{f}$ and $f_{r+1}$ are as in (5.2.7), $h \in k[w_1, \dots, w_{M-1}]$ is an arbitrary polynomial of degree $d-2$, and $\delta_{M,2}$ is the Kronecker delta, i.e. $\delta_{M,2} = 1$ if $M = 2$ and $\delta_{M,2} = 0$ otherwise.

We check that the polynomials $\tilde{g}$ and $g_{r+1}$ in (5.2.8) satisfy the conditions in the claim by using some abstract field isomorphism $k \simeq \overline{k(t)}$, see also Step 1.

The admissibility of $\tilde{g}$ and $g_{r+1}$ with respect to $B \otimes_k \overline{k(t)}[w_1, \dots, w_M]$ and condition ($\star$) can be shown via similar arguments as in Section 5.2. We provide some details for the convenience of the reader. Identify $\overline{k(t)}$ with $\overline{k(t, t_2)}$ and consider

$$A \coloneqq B \otimes_k k(t, t_2)[w_1, \dots, w_M].$$

Note that $A$ is strongly $k(t, t_2)$-rational, as $B$ is strongly $k$-rational by (C1), see Example 5.2.2 (3); in particular $A$ is a smooth geometrically integral $k(t, t_2)$-algebra of finite type by Lemma 5.2.4. Viewing $\tilde{g}$ and $g_{r+1}$ from (5.2.8) as elements in $A[z]$, we find that

$$\begin{aligned} A[z]/(\tilde{g}) &\simeq B[w_1, \dots, w_d, z]/(\tilde{f}) \otimes_k k(t, t_2) \\ A/(g_{r+1}) &\overset{(5.2.8)}{=} A/\left(t_2 - w_M(t_1 - w_{M-1}g - t(w_{M-1} - \delta_{M,2})h)\right) \end{aligned}$$

are strongly $k(t, t_2)$-rational by the definition of admissible (Definition 5.2.5) for

$\tilde{f}$ and $f_{r+1}$ and Example 5.2.2 (2). The elements $\tilde{g} \in A[z]$ and $g_{r+1} \in A$ satisfy condition ($\star$), as for every field extension $F/k(t,t_2)$ and any $q_1, q_2 \in F$, we have an $F$-algebra epimorphism

$$\frac{A[z]_{\partial_z \tilde{g}} \otimes_{k(t,t_2)} F}{(\tilde{g}+q_1, g_{r+1}+q_2)} \twoheadrightarrow \frac{B \otimes_k F[w_1, \ldots, w_{M-2}, z]_{\partial_z f}}{\left(f + w_1 + \cdots + w_{M-2} + \delta_{M,2} + \frac{t_2+q_2}{t_1-\delta_{M,2}g} + q_1\right)} \overset{(C4)}{\twoheadrightarrow} F. \quad (\dagger)$$

The first arrow is the quotient by the ideal $(w_{M-1} - \delta_{M,2})$ together with an isomorphism of $F$-algebras using the explict description of $\tilde{g}$ and $g_{r+1}$ from (5.2.8). This uses that $t_1 - \delta_{M,2} g \in k^*$. In particular, we see that the $k(t,t_2)$-scheme

$$\operatorname{Spec} A[z]_{\partial_z \tilde{g}}/(\tilde{g}, g_{r+1}) = \operatorname{Spec}\left(A/(g_{r+1})\right)[z]_{\partial_z \tilde{g}}/(\tilde{g}) \tag{5.2.9}$$

contains a $k(t,t_2)$-rational point and is smooth over $k(t,t_2)$, as it is a standard smooth $\operatorname{Spec} A/(g_{r+1})$-scheme and the $k(t,t_2)$-algebra $A/(g_{r+1})$ is strongly $k(t,t_2)$-rational by the above, see also Lemma 5.2.4. The scheme (5.2.9) is an open subscheme of $\operatorname{Spec} A[z]/(\tilde{g}, g_{r+1})$, see e.g. the argument in Remark 5.2.3. Both schemes are integral $k(t,t_2)$-varieties, as $A[z]/(\tilde{g}, g_{r+1})$ is a localization of the integral domain

$$B \otimes_k k(t)[t_2, w_1, \ldots, w_M, z]/(\tilde{g}, g_{r+1}) \overset{(5.2.8)}{\simeq} B \otimes_k k(t)[w_1, \ldots, w_M, z]/(\tilde{f}). \tag{5.2.10}$$

Thus, the $k(t,t_2)$-scheme (5.2.9) is geometrically integral by [Stacks, Tag 0CDW]. Since (5.2.9) is an open subscheme of the $k(t,t_2)$-variety $\operatorname{Spec} A[z]/(\tilde{g}, g_{r+1})$, which is also Cohen-Macaulay by [Eis95, Proposition 18.13], the variety $\operatorname{Spec} A[z]/(\tilde{g}, g_{r+1})$ is also geometrically integral over $k(t,t_2)$. Hence $\tilde{g}$ and $g_{r+1}$ are admissible with respect to $A$, in particular also with respect to $B \otimes_k \overline{k(t)}[w_1, \ldots, w_M]$.

The statement about the torsion order follows now directly from Lemma 4.2.9 via the degeneration $t \to 0$. Hence, the polynomials $\tilde{g}$ and $g_{r+1}$ satisfy the conditions in the claim for $d > M \geq 2$, which concludes Step 2 and the proof of the claim.

We turn to the construction of $\check{f}$ and the proof of (b). Let $d$ and $M$ be positive integers such that $d \geq M$. Then, the claim applied for the integers $d+1$ and $M+1$ shows that there exists polynomials

$$\tilde{f} \in k[x_1, \ldots, x_{n+r}, w_1, \ldots, w_{M+1}, z] \quad \text{and} \quad f_{r+1} \in k[w_1, \ldots, w_{M+1}]$$

of degree $\deg f$ and $d+1$, respectively, such that the conditions (C1)–(C4) are satisfied for $f_1, \ldots, f_{r+1}$ and $\tilde{f}$. In fact, the construction in Step 2 shows that the polynomials

$$\begin{aligned}\tilde{f} &= f + w_1 + \cdots + w_{M+1} \in k[x_1, \dots, x_{n+r}, w_1, \dots, w_{M+1}, z] \text{ and}\\ f_{r+1} &= t_2 - w_{M+1}\,(t_1 - w_M g - t(w_M - \delta_{M+1,2})h) \in k[w_1, \dots, w_{M+1}]\end{aligned}$$

work for some specific $t, t_2 \in k^*$ and $t_1 \in k$ and some polynomials $g \in k[w_1, \dots, w_{M-1}]$ and $h \in k[w_1, \dots, w_M]$ of degree $M-1$ and $d-1$, respectively, see (5.2.7) and (5.2.8). We note that there is a $B$-algebra isomorphism

$$B[w_1, \dots, w_{M+1}, z]/(\tilde{f}, f_{r+1}) \simeq B[w_1, \dots, w_M, z]/(\check{f}), \tag{5.2.11}$$

where $\check{f}$ is the degree $(d + \deg f)$ polynomial

$$\check{f} \coloneqq t_2 + (f + w_1 + \cdots + w_M)(t_1 - w_M g - t(w_M - \delta_{M+1,2})h) \tag{5.2.12}$$

in $k[x_1, \dots, x_{n+r}, w_1, \dots, w_M, z]$. It is straightforward to see from the isomorphism (5.2.11) and the explicit form of $\check{f}$ in (5.2.12) that (b) holds for $\check{f}$ as in (5.2.12). □

**Remark 5.2.13.** (a) The proof shows that the polynomial $l$ in (C3) for the polynomials $f_1, \dots, f_r$ and $f$ also works for $f_1, \dots, f_{r+1}$ and $\tilde{f}$ as well as for $f_1, \dots, f_r$ and $\check{f}$.

(b) The proof shows that the polynomial $\tilde{f}$ can be choosen as $\tilde{f} = f + h$ for some linear polynomial $h \in k[w_1, \dots, w_M]$, see the claim in the above proof.

(c) If $M = 2$, then we can also choose the polynomials in (5.2.8) as

$$\tilde{g} \coloneqq \tilde{f} = f + w_1 + w_2 \quad \text{and} \quad g_{r+1} \coloneqq t_2 - w_1 w_2 + t w_1^d.$$

Indeed, the proof uses the strongly $k(t, t_2)$-rationality of $A/(g_{r+1})$, the existence of the epimorphism (†), and the isomorphism (5.2.10) as well as that $g_{r+1}$ specializes to $f_{r+1}$ from (5.2.7) via $t \to 0$. The last two properties are obviously still satisfied for the new choice of $g_{r+1}$. We can replace (†) by

$$A[z]_{\partial_z \tilde{g}} \otimes_{k(t,t_2)} F/(\tilde{g}+q_1, g_{r+1}+q_2) \twoheadrightarrow B \otimes_k F[z]_{\partial_z f}/\left(f + t_2 + t + q_2 + q_1 + 1\right) \twoheadrightarrow F$$

where the first map is the quotient by the ideal $(w_1 - 1)$ and the second map exists by (C4). The strongly $k(t, t_2)$-rationality of $A/(g_{r+1})$ follows from

$$A/(g_{r+1}) \simeq B \otimes_k k(t, t_2)[w_1, w_2]/(t_2 - w_1(w_2 + t w_1^{d-1})) \simeq B \otimes_k k(t, t_2)[w_1]_{w_1}.$$

## 5.3. Base examples

To apply the machinery of affine degenerations from Section 5.2, we need explicit equations of (rationally connected) hypersurfaces with no integral decomposition of the diagonal.

### 5.3.1. Schreieder's hypersurface examples

In [Sch19b, Sch21a], Schreieder constructs examples of singular hypersurfaces with non-trivial unramified cohomology classes, in particular these examples have no decomposition of the diagonal. Starting from these explicit hypersurfaces, Schreieder and the first named author [LS25] manage to increase the dimension of the hypersurface examples via a degeneration argument, see Theorem 4.3.8. In this subsection, we recall these constructions from Chapter 4 and show that they fit into our affine degeneration framework, leading to the following result.

**Theorem 5.3.1.** *Let $k$ be an algebraically closed and uncountable field. Let $n', m \geq 2$ be integers such that $m$ is invertible in $k$. Then for every integer $N$ satisfying*

$$3 \leq N \leq n' + 2^{n'} - 2 + \sum_{l=0}^{n'-1} \binom{n'}{l} \left\lfloor \frac{l}{m} \right\rfloor$$

*and any integer $d \geq n' + m$, there exists an irreducible polynomial*

$$f \in k[x_1, \dots, x_N, z]$$

*of degree $d$ satisfying condition ($\star\star$) and such that*

$$\operatorname{Tor}^{\mathbb{Z}/m}(\operatorname{Spec} k[x_1, \dots, x_N, z]/(f), \operatorname{Spec} k[x_1, \dots, x_N, z]/(f, \partial_z f)) = m. \tag{5.3.1}$$

*In particular, $f$ satisfies (C1)–(C4) in Theorem 5.2.12 for $n = N$, $r = 0$.*

We start with the singular hypersurface examples in [Sch21a], see also Section 4.5.

**Example 5.3.2.** Let $k$ be an algebraically closed field containing an element $\pi \in k$, which is transcendental over the prime field of $k$. Fix integers $m, n \geq 2$ such that $m$ is invertible in $k$ and consider the polynomial

$$g \coloneqq \pi \cdot \left(1 + \sum_{i=1}^{n} x_i^{\lceil \frac{n+1}{m} \rceil}\right)^m - (-1)^n x_1 x_2 \dots x_n \in k[x_1, \dots, x_n]. \tag{5.3.2}$$

For an integer $N$ satisfying $n+1 \leq N \leq n+2^n-2$, we consider the polynomial

$$f_0 \coloneqq g(x_1,\dots,x_n) + \sum_{j=1}^{N-n} c_j(x_1,\dots,x_n)x_{n+j}^m + (-1)^n x_1 x_2 \dots x_n z^m \tag{5.3.3}$$

in $k[x_1,\dots,x_N,z]$, where $g$ is as in (5.3.2) and $c_j \coloneqq (-x_1)^{\varepsilon_1}(-x_2)^{\varepsilon_2}\dots(-x_n)^{\varepsilon_n}$ in $k[x_1,\dots,x_n]$ for the unique $\varepsilon_i \in \{0,1\}$ satisfying $j = \sum_{i=1}^n \varepsilon_i 2^{i-1}$. We note that $\deg f_0 = n+m$ and that $c_1(x_1,\dots,x_n) = x_1$.

In fact, $f_0$ is the dehomogenization of the homogeneous polynomial $F$ appearing in (4.5.3). Thus, Theorem 4.5.1 implies that $f_0$ is an irreducible polynomial and

$$\mathrm{Tor}^{\mathbb{Z}/m}(\operatorname{Spec} k[x_1,\dots,x_N,z]/(f_0), \operatorname{Spec} k[x_1,\dots,x_N,z]/(f_0,\partial_z f_0)) = m,$$

as $\partial_z f_0 = (-1)^n m x_1 \cdots x_n z^{m-1}$. In particular, $f_0$ satisfies the conditions (C1)–(C3) in Theorem 5.2.12 for $n=N$ and $r=0$; the lemma below shows condition (C4).

**Lemma 5.3.3.** *Let the notation be as in Example 5.3.2. Then for any field extension $F/k$ and any element $q \in F$, there exists an $F$-algebra epimorphism*

$$F[x_1,\dots,x_N,z,w]/(f_0+q, (-1)^n m x_1 \cdots x_n z^{m-1} w - 1) \twoheadrightarrow F. \tag{5.3.4}$$

*Proof.* Let $F/k$ be a field extension and let $q \in F$. It clearly suffices to construct the epimorphism in the case that $N = n+1$. If $q \in F$ is defined over $k$, then the existence of an epimorphism (5.3.4) is straightforward, as $k$ is algebraically closed. Indeed, there exists clearly an $k$-algebra epimorphism

$$k[x_1,\dots,x_{n+1},z,w]/(f_0+q, w\partial_z f_0 - 1) \twoheadrightarrow k[x_{n+1}]/(x_{n+1}^m + \pi(n+1)^m + q) \twoheadrightarrow k,$$

where the first morphism is the quotient by the ideal $(x_1-1,\dots,x_n-1,z-1)$ and the second homomorphism exists as $k$ is algebraically closed. Since tensor product is right-exact, the $F$-algebra surjection (5.3.4) exists for $q \in k$.

Thus, we may assume now that $q \in F \setminus k$. Let $\zeta \in k$ be such that $\zeta^{\lceil \frac{n+1}{m} \rceil} = -1$, which exists as $k$ is algebraically closed. Then the quotient by the ideal

$$(x_2 - \zeta x_1, x_3 - 1, \dots, x_{n+1} - 1, z-1) \subset F[x_1,\dots,x_{n+1},z,w]$$

yields an $F$-algebra homomorphism

$$F[x_1,\dots,x_{n+1},z]_{\partial_z f_0}/(f_0+q) \twoheadrightarrow F[x_1]_{x_1}/(\pi(n-1)^m + x_1 + q) \simeq F.$$

where we used for the isomorphism that $\pi(n-1)^m + q \neq 0$, as $q \notin k$. □

Starting from the above examples, we constructed in Section 4.4 a series of hypersurface examples without a decomposition of the diagonal via the "double cone construction", see also [Moe23]. We state an affine version of this construction.

**Example 5.3.4.** Let $k$ be an algebraically closed field and let $N, m, d \in \mathbb{Z}$ be positive integers such that $N, m \geq 2$, $m$ is invertible in $k$, and $2m \leq d$. Suppose there exists an irreducible polynomial $f \in k[x_1, \ldots, x_N, z]$ of the form

$$f \coloneqq bz^m + \sum_{i=0}^{m} a_i x_{j_0}^i \tag{5.3.5}$$

for some index $j_0 \in \{1, \ldots, N\}$ and some polynomials

$$b, a_0, \ldots, a_m \in k[x_1, \ldots, \widehat{x_{j_0}}, \ldots, x_N]$$

not containing the variable $x_{j_0}$ and of degree $\deg b = d - m$, $\deg a_i \leq d - i$ such that

$$\operatorname{Tor}^{\mathbb{Z}/m}(\operatorname{Spec} k[x_1, \ldots, x_N, z]/(f), \operatorname{Spec} k[x_1, \ldots, x_N, z]/(f, \partial_z f)) = m.$$

For example, the polynomial $f_0$ in (5.3.3) satisfies the assumptions for $j_0 = n + 1$.

Let $K/k$ be an algebraically closed field extension of transcendence degree 2 and let $\lambda, t$ denote a transcendental basis of $K/k$. Consider the $K$-scheme

$$X \coloneqq \{f(x_1, \ldots, x_N, z) + w_0 + (\lambda x_{j_0} + 1)w_1 = t - w_0 w_1 = 0\} \subset \mathbb{A}_K^{N+3},$$

where $x_1, \ldots, x_N, z, w_0, w_1$ are the coordinates of $\mathbb{A}_K^{N+3}$. Note that $X$ is as a $K$-scheme isomorphic to

$$\begin{aligned} X &\simeq \operatorname{Spec} K[x_1, \ldots, x_N, z, w_0, w_0^{-1}]/(f(x_1, \ldots, x_N, z) + w_0 + (\lambda x_{j_0} + 1)tw_0^{-1}) \\ &\simeq \operatorname{Spec} K[x_1, \ldots, x_N, z, w_0, w_0^{-1}]/\left(bz^m + \sum_{i=0}^{m} a_i(w_0 x_{j_0} - \lambda^{-1})^i + w_0 + t\lambda x_{j_0}\right), \end{aligned}$$

where the second isomorphism is given by the transformation $x_{j_0} \mapsto (w_0 x_{j_0} - \lambda^{-1})$.

If $\deg a_i \leq d - 2i$ for all $i \in \{0, \ldots, m\}$, then $X$ is isomorphic to an open subscheme of the affine degree $d$ hypersurface in $\mathbb{A}_K^{N+2}$ given by the irreducible polynomial

$$\tilde{f} \coloneqq bz^m + \sum_{i=0}^{m} a_i(w_0 x_{j_0} - \lambda^{-1})^i + w_0 + t\lambda x_{j_0} \in K[x_1, \ldots, x_N, z, w_0]. \tag{5.3.6}$$

By degenerating $X$ via $t \to 0$, it follows from Proposition 4.4.7 that

$$\mathrm{Tor}^{\mathbb{Z}/m}\left(\operatorname{Spec} K[x_1,\dots,x_N,z,w_0]/(\tilde{f}), \operatorname{Spec} K[x_1,\dots,x_N,z,w_0]/(\tilde{f}, w_0 \cdot \partial_z \tilde{f})\right) = m,$$

see also the proof of Theorem 4.6.1. Note that $\tilde{f}$ is again of the form (5.3.5).

Observe that $\tilde{f}$ automatically satisfies the condition ($\star\star$). Indeed, since $b$ is a non-zero polynomial over an infinite field $k$, there exists $\alpha \coloneqq (\alpha_i)_{i=1,\dots,\hat{j}_0,\dots,N} \in k^{N-1}$ such that $b(\alpha) \neq 0$. Then the quotient by the ideal

$$(z-1, x_{j_0}, x_i - \alpha_i : i \neq j_0)$$

induces for any field extension $F/K$ and any $q \in F$ the following $F$-algebra epimorphism

$$F[x_1,\dots,x_N,z,w_0,w]/(\tilde{f}+q, w\partial_z\tilde{f}-1) \twoheadrightarrow F[w_0]/\left(f(\alpha_1,\dots,\alpha_N,1)+w_0+q\right) \simeq F,$$

where we set $\alpha_{j_0} \coloneqq -\lambda$.

Repeatedly applying the "double cone construction" to the polynomial $f_0$ in (5.3.3) yields polynomials of degree $\deg f_0$ in a polynomial ring of larger Krull dimension.

**Remark 5.3.5.** Let $n, m \geq 2$ be integers such that $n, m \geq 2$. Then for $N = n+2^n-2$ the polynomial $f_0$ from Equation (5.3.3) reads

$$f_0 = g(x_1,\dots,x_n) + \sum_{j=1}^{2^n-2} c_j(x_1,\dots,x_n)x_{n+j}^m + (-1)^n x_1x_2\dots x_n z^m \tag{5.3.7}$$

in $k[x_1,\dots,x_{n+2^n-2},z]$, where $g, c_1,\dots,c_{2^n-2} \in k[x_1,\dots,x_n]$ are some specific non-zero polynomials, see Example 5.3.2. In particular, we see that the "double cone construction" is applicable for $j_0 \in \{n+1, n+2,\dots,n+2^n-2\}$.

A quick analysis of the change of the polynomials $a_i$ under the "double cone construction" $f \to \tilde{f}$ shows that we can apply the construction from Example 5.3.4 in total

$$\sum_{j=1}^{2^n-2}\left\lfloor\frac{n-\deg c_j}{m}\right\rfloor = \sum_{l=0}^{n-1}\binom{n}{l}\left\lfloor\frac{l}{m}\right\rfloor$$

times, see the proof of Theorem 4.6.1 and also Lemma 4.4.2.

We summarize the results of the above examples (Example 5.3.2 and Example 5.3.4) before turning to the proof of the Theorem 5.3.1. Let $k$ be an algebraically closed and uncountable field and let $n, m \geq 2$ be integers such that $m$ is invertible in $k^*$. For every integer $N$ with $n+1 \leq N \leq n+2^n-2$, there exists a polynomial

$$f \in k[x_1, \dots, x_N, z] \tag{5.3.8}$$

which is irreducible of degree $n+m$ and satisfies the conditions ($\star\star$) and (5.3.1).

*Proof of Theorem 5.3.1:* The construction of the polynomial $f$ involves the existence of a polynomial in (5.3.8) together with Theorem 5.2.12 (b). Let $N, d$ be integers satisfying

$$d \geq n' + m \quad \text{and} \quad 4 \leq N \leq n' + 2^{n'} - 2 + \sum_{l=0}^{n'-1} \binom{n'}{l} \left\lfloor \frac{l}{m} \right\rfloor.$$

Then there exists an integer $2 \leq n \leq n'$ such that

$$n + 1 \leq N - 1 < N \leq n + 2^n - 2 + \sum_{l=0}^{n-1} \binom{n}{l} \left\lfloor \frac{l}{m} \right\rfloor.$$

Note that $d \geq n' + m \geq n + m$. If $d = n + m$, then there exists by (5.3.8) an irreducible polynomial $f \in k[x_1, \dots, x_N, z]$ of degree $d$ satisfying condition ($\star\star$) and (5.3.1). So, we can assume $d > n + m$, then there exists by (5.3.8) an irreducible polynomial $\tilde{f} \in k[x_1, \dots, x_{N-1}, z]$ of degree $n+m$ satisfying condition ($\star\star$) and

$$\mathrm{Tor}^{\mathbb{Z}/m}\left(\operatorname{Spec} k[x_1, \dots, x_{N-1}, z]/(\tilde{f}), \operatorname{Spec} k[x_1, \dots, x_{N-1}, z]/(\tilde{f}, \partial_z \tilde{f})\right) = m.$$

Note that this uses Lemma 4.2.6 (b) and (e). Applying Theorem 5.2.12 (b) to the polynomial $\tilde{f}$ with $r = 0$ and added degree $d-n-m$ and 1 added variable, there exists an irreducible polynomial $f \in k[x_1, \dots, x_N, z]$, denoted by $\check{f}$ in Theorem 5.2.12, of degree $d = n + m + (d - n - m)$ satisfying condition ($\star\star$) and such that

$$\mathrm{Tor}^{\mathbb{Z}/m}\left(\operatorname{Spec} k[x_1, \dots, x_N, z]/(f), \operatorname{Spec} k[x_1, \dots, x_N, z]/(f, \partial_z f)\right) = m,$$

i.e. the polynomial $f$ satisfies the claims in the theorem.

It remains to show the theorem for $N = 3$. Since $k$ is an uncountable field, there exists $\pi, \rho \in k$ which are algebraically independent over the prime field of $k$. We claim that for any $d \geq 2 + m$ the polynomial

$$f := \rho x_3^d + \pi \left(1 + x_1^{\lceil \frac{3}{m} \rceil} + x_2^{\lceil \frac{3}{m} \rceil}\right)^m - x_1 x_2 + x_1 x_3^m + x_1 x_2 z^m \in k[x_1, x_2, x_3, z]$$

satisfies the properties stated in the theorem. Indeed, a direct computation shows that if $f \in k[x_1, x_2, x_3][z]$ would be reducible, then it has to be divisible by $x_1x_2$, which it is clearly not. Thus $f$ is irreducible. We note that $f$ degenerates via $\rho \to 0$ to a polynomial $f_0$ of the form (5.3.3), thus it follows from Lemma 4.2.9 that

$$\mathrm{Tor}^{\mathbb{Z}/m}\left(\operatorname{Spec} k[x_1, x_2, x_3, z]/(f), \operatorname{Spec} k[x_1, x_2, x_3, z]/(f, \partial_z f)\right) = m,$$

see also [LS25, Corollary 6.2]. The condition ($\star\star$) follows by a similar argument as in Lemma 5.3.3, which shows the claim and thus the theorem. $\square$

### 5.3.2. A special quartic fourfold

We provide three examples of affine complete intersections, namely a quartic fourfold, a $(2, 2, 2)$-fourfold, and a $(3, 3)$-fivefold, which also fit in our framework of affine degenerations from Section 5.2, which are used in the proof of Theorem 5.1.2 for the case of small Fano index $r \leq 2$, see also Proposition 5.4.3. These are based on a special quartic fourfold example from [LS25, Example 4.6], which is birational to the quadric surface bundle example in [HPT18a]. Throughout this section, let $k$ be an algebraically closed field of characteristic $\operatorname{char} k \neq 2$.

**Example 5.3.6.** Consider the irreducible polynomial

$$h := y_1^2 + x_1x_2y_2^2 + x_2y_3^2 + x_1(1 + x_1^2 + x_2^2 - 2x_1 - 2x_2 - 2x_1x_2) \in k[x_1, x_2, y_1, y_2, y_3].$$

The argument sketched in Example 4.3.6 shows that the $\mathbb{Z}/2$-torsion order of the affine scheme

$$X := \operatorname{Spec} k[x_1, x_2, y_1, y_2, y_3]/(h)$$

relative to the closed subscheme $W := \{x_1x_2y_1 = 0\} \subset X$ is equal to 2. In fact, the argument presented there can be adapted[3] to show that

$$\mathrm{Tor}^{\mathbb{Z}/2}\left(\operatorname{Spec} k[x_1, x_2, y_1, y_2, y_3]/(h), \operatorname{Spec} k[x_1, x_2, y_1, y_2, y_3]/(h, x_1x_2y_1y_2y_3)\right) = 2.$$

We would like to emphasize that this relies on some vanishing results for unramified cohomology classes, for instance [Sch19b, Theorem 9.2].

By replacing $y_1 = x_1z_1$, $y_2 = z_2$, and $y_3 = x_1z_3$ and dividing the resulting polynomial by $x_1$, we find that $X$ is birational to the affine hypersurface associated

[3] the polynomial $g$ in Example 4.3.6 is invariant under any permutation of the coordinates $x_0, x_1, x_2$.

to the following irreducible polynomial in $A \coloneqq k[x_1, x_2, z_1, z_2, z_3]$:

$$f \coloneqq x_1 z_1^2 + x_2 z_2^2 + x_1 x_2 z_3^2 + (1 + x_1^2 + x_2^2 - 2x_1 - 2x_2 - 2x_1 x_2). \tag{5.3.9}$$

In particular, we see from the above explicit change of coordinates that

$$\mathrm{Tor}^{\mathbb{Z}/2}(\operatorname{Spec} A/(f), \operatorname{Spec} A/(f, x_1 x_2 z_1 z_2 z_3)) = 2. \tag{5.3.10}$$

To see that the polynomial $f$ fits in our framework of affine degeneration, we check the condition ($\star\star$) by a similar argument as in the proof of Lemma 5.3.3.

**Lemma 5.3.7.** *For any field extension $F/k$ and any $q \in F$, there exists an $F$-algebra epimorphism*

$$F[x_1, x_2, z_1, z_2, z_3, w]/(f + q, w\partial_{z_3} f - 1) \twoheadrightarrow F,$$

*where $f \in k[x_1, x_2, z_1, z_2, z_3]$ is the polynomial in* (5.3.9).

*Proof.* Let $F/k$ be a field extension and let $q \in F$. If $q$ is defined over $k$, then there is a $k$-algebra epimorphism

$$k[x_1, x_2, z_1, z_2, z_3, w]/(f + q, 2wx_1x_2z_3 - 1) \twoheadrightarrow k[z_1, z_2]/(z_1^2 + z_2^2 - 2 + q) \twoheadrightarrow k,$$

where the first morphism is the quotient by the ideal $(x_1 - 1, x_2 - 1, z_3 - 1)$ and the second morphism exists as $k$ is algebraically closed. Thus the lemma follows for $q \in k \subset F$ by the right-exactness of the tensor product. Thus, we may assume $q \in F \setminus k$. Then, the quotient by the ideal $(x_1 + x_2 - 1, z_1 - 1, z_2, z_3 - 2)$ yields the claimed $F$-algebra epimorphism

$$F[x_1, x_2, z_1, z_2, z_3, w]/(f+q, 2wx_1x_2z_3-1) \twoheadrightarrow F[x_1, w]/(x_1+q, 4wx_1(x_1-1)+1) \simeq F,$$

where we used that $q \neq 0, 1$. □

We recall the explicit birational equivalence of the (affine) quartic fourfold in Example 5.3.6 with a $(2, 2, 2)$-complete intersection from [HPT18b, §3.1], which is based on a construction of Beauville [Bea77].

**Example 5.3.8.** Consider the polynomials

$$p_1 \coloneqq -y_1 + z_2^2 + z_3 z_4 - 2, \quad p_2 \coloneqq y_1 + y_2 z_3 + z_1^2 - 2, \quad p_3 \coloneqq y_1 z_5 - y_2 z_4 + 1 + z_5^2 \tag{5.3.11}$$

in $k[y_1, y_2, z_1, z_2, z_3, z_4, z_5]$, see [HPT18b, Equation (3.3)]. Then there exists an isomorphism of $k$-algebras

$$k[y_1, y_2, z_1, z_2, z_3, z_4, z_5]_{z_3}/(p_1, p_2, p_3) \simeq k[x_1, x_2, z_1, z_2, z_3]_{z_3}/(f),$$

where $f \in k[x_1, x_2, z_1, z_2, z_3]$ is as in (5.3.9). In particular, the affine $(2, 2, 2)$-complete intersection $\{p_1 = p_2 = p_3 = 0\} \subset \mathbb{A}^7_k$ is birational to the affine quartic fourfold $\{f = 0\} \subset \mathbb{A}^5_k$. Indeed, the following sequence of isomorphism provides the claimed isomorphism:

$$\begin{aligned}
&k[y_1, y_2, z_1, z_2, z_3, z_4, z_5]_{z_3}/(-y_1 + z_2^2 + z_3z_4 - 2, y_1 + y_2z_3 + z_1^2 - 2, y_1z_5 - y_2z_4 + 1 + z_5^2)\\
\simeq\ & k[z_1, z_2, z_3, z_4, z_5]_{z_3}/\left((z_2^2 + z_3z_4 - 2)z_5 + (z_2^2 + z_3z_4 + z_1^2 - 4)z_3^{-1}z_4 + 1 + z_5^2\right)\\
=\ & k[z_1, z_2, z_3, z_4, z_5]_{z_3}/\left(z_3^{-1}z_4z_1^2 + (z_5 + z_3^{-1}z_4)z_2^2 + z_3z_4z_5 + z_4^2 + 1 + z_5^2 - 2z_5 - 4z_3^{-1}z_4\right)\\
\simeq\ & k[x_1, x_2, z_1, \dots, z_5]_{z_3}/\left(x_1 - z_3^{-1}z_4, x_2 - x_1 - z_5, x_1z_1^2 + x_2z_2^2 + x_2z_3z_4 + (z_5 - 1)^2 - 4x_1\right)\\
\simeq\ & k[x_1, x_2, z_1, z_2, z_3]_{z_3}/\left(x_1z_1^2 + x_2z_2^2 + x_1x_2z_3^2 + (x_2 - x_1 - 1)^2 - 4x_1\right).
\end{aligned}$$

We could not verify the condition (C1) for any two of the three $p_i$'s. It seems that the polynomials $p_1, p_2, p_3$ in (5.3.11) do not fit directly in the setup of Theorem 5.2.12. Instead, we consider a different affine chart of the projective closure of $\{p_1 = p_2 = p_3 = 0\} \subset \mathbb{A}^7$.

**Example 5.3.9.** Consider the $k$-algebra $k[y_1, y_2, z_1, z_2, z_3, z_4, z_5]$ and the polynomials $p_1, p_2, p_3$ from (5.3.11). We perform the following change of coordinates

$$(x_1, \dots, x_5, x_6, x_7) \coloneqq (y_2z_4^{-1}, z_1z_4^{-1}, z_2z_4^{-1}, z_3z_4^{-1}, z_4^{-1}, y_1z_4^{-1}, z_5z_4^{-1}).$$

Then the quadric polynomials $p_1, p_2, p_3$ from (5.3.11) read

$$\begin{aligned}
q_1 &\coloneqq -x_6x_5 + x_3^2 + x_4 - 2x_5^2 \in k[x_1, \dots, x_7],\\
q_2 &\coloneqq x_6x_5 + x_1x_4 + x_2^2 - 2x_5^2 \in k[x_1, \dots, x_7],\\
q_3 &\coloneqq x_6x_7 - x_1 + x_5^2 + x_7^2 \in k[x_1, \dots, x_7].
\end{aligned} \tag{5.3.12}$$

The above constructions imply that the affine $(2, 2, 2)$-complete intersection given by $q_1, q_2, q_3$ satisfies the assumptions in Theorem 5.2.12 for $m = 2$.

**Corollary 5.3.10.** *For $q_1, q_2, q_3 \in k[x_1, \dots, x_7]$ as in (5.3.12) the following holds:*

*(i) The $k$-algebra $k[x_1, x_3, x_4, x_5, x_6, x_7]/(q_1, q_3) \simeq k[x_3, x_5, x_6, x_7]$ is strongly $k$-rational;*

*(ii)* $X \coloneqq \operatorname{Spec} k[x_1,\dots,x_7]/(q_1,q_2,q_3)$ *is an integral $k$-variety of dimension* 4.

*(iii)* *Set* $W \coloneqq \{x_2x_4x_5 = 0\} \subset X$*, then* $\operatorname{Tor}^{\mathbb{Z}/2}(X,W) = 2$*;*

*(iv)* *For any field extension $F/k$ and $q \in F$, there exists an $F$-algebra epimorphism*

$$F[x_1,\dots,x_7,w]/(q_1,q_2+q,q_3,w\partial_{x_2}q_2-1) \twoheadrightarrow F.$$

*Proof.* Statement (i) is obvious and statement (ii) follows from [HPT18b]. The isomorphisms in Example 5.3.8 and the change of coordinates in Example 5.3.9 imply that

$$\begin{aligned} k[x_1,x_2,z_1,z_2,z_3]_{x_1z_3}/(f) &\longrightarrow k[x_1,\dots,x_7]_{x_4x_5}/(q_1,q_2,q_3),\\ (x_1,x_2,z_1,z_2,z_3) &\longmapsto (x_4^{-1},x_7x_5^{-1}+x_4^{-1},x_2x_5^{-1},x_3x_5^{-1},x_4x_5^{-1}) \end{aligned}$$

is an isomorphism of $k$-algebras, where $f$ is as in (5.3.9). Via this isomorphism statement (iii) follows from (5.3.10). We turn to the proof of statement (iv). Let $F/k$ be a field extension and let $q \in F$. Note first $F[x_1\dots,x_7]/(q_1,q_2+q,q_3)$ is equal to

$$F[x_2,x_3,x_5,x_6,x_7]/(x_6x_5+(x_5^2+x_7^2+x_6x_7)(2x_5^2+x_6x_5-x_3^2)+x_2^2-2x_5^2+q).$$

Thus the quotient by the ideal $(x_2-1,x_3-1,x_5,x_7-1)$ induces the $F$-algebra epimorphism

$$F[x_1,\dots,x_7,w]/(q_1,q_2+q,q_3,2x_2w-1) \twoheadrightarrow F[x_6]/(-x_6+q) \simeq F,$$

which shows statement (iv) and thus completes the proof of the corollary. $\square$

We conclude this section by constructing an affine $(3,3)$-fivefold whose relative $\mathbb{Z}/2$-torsion order is divisible by 2. By applying the results from Section 5.2 to an affine $(2,3)$-complete intersection isomorphic to the affine quartic fourfold in Example 5.3.6.

**Example 5.3.11.** From the explicit form of the polynomial $f$ in (5.3.9), we see that there exists an $k$-algebra isomorphism

$$k[x_1,x_2,z_1,z_2,z_3]/(f) \simeq k[x_1,x_2,x_3,z_1,z_2,z_3]/(f_1,f_2)$$

where $f_1, f_2 \in k[x_1, x_2, x_3, z_1, z_2, z_3]$ are the polynomials

$$f_1 \coloneqq x_3 - z_3^2, \quad f_2 \coloneqq x_1 z_1^2 + x_2 z_2^2 + x_1 x_2 x_3 + (1 + x_1^2 + x_2^2 - 2x_1 - 2x_2 - 2x_1 x_2). \quad (5.3.13)$$

Note that the $k$-algebra $B \coloneqq k[x_1, x_2, x_3, z_1, z_2]_{x_1 x_2}/(f_2)$ is strongly $k$-rational and that $B[z_3]/(f_1)$ is a geometrically integral $k$-algebra of Krull dimension 4, as it is a localization of a geometrically integral $k$-algebra. Moreover, we have for any field extension $F/k$ and any element $q \in F$ an $F$-algebra epimorphism

$$B \otimes_k F[z_3]_{\partial_{z_3} f_1}/(f_1 + q) = k[x_1, x_2, x_3, z_1, z_2, z_3]_{x_1 x_2 z_3}/(f_1 + q, f_2) \twoheadrightarrow F \quad (5.3.14)$$

given by the ideal $(z_3 - \alpha, x_2 - x_3, x_1 - x_2 - 1, z_1 - \sqrt{-1} x_3, z_2 - 2)$, where $\alpha \in k^*$ is chosen such that $\alpha^2 + q \neq -1, 0$. In particular, we find that $f_1 + x_4 \in B[z_3, x_4]$ and $x_4 \in B[x_4]$ are admissible with respect to $B[x_4]$ and satisfy condition ($\star$) by Example 5.2.10.

Thus, we can apply Theorem 5.2.11 and find by (5.3.10) that the affine scheme

$$X' \coloneqq \operatorname{Spec} k(t)[x_1, x_2, x_3, x_4, z_1, z_2, z_3]_{x_1 x_2}/(t - x_4(f_1 + x_4), f_2),$$

is a geometrically integral $k(t)$-variety of dimension 5 satisfying

$$\operatorname{Tor}^{\mathbb{Z}/2}(X' \times_{k(t)} \overline{k(t)}, W' \times_{k(t)} \overline{k(t)}) = 2,$$

where $W' \coloneqq \{x_1 x_2 z_1 z_3 = 0\} \subset X'$ and $k(t)/k$ is a purely transcendental field extension of transcendence degree 1. In fact, the $k(t)$-scheme

$$X \coloneqq \operatorname{Spec} k(t)[x_1, x_2, x_3, x_4, z_1, z_2, z_3]/(t - x_4(f_1 + x_4), f_2)$$

is an integral $k$-variety by a similar argument as in Proposition 5.2.6 and hence of dimension 5. Moreover, $X$ is Cohen-Macaulay by [Eis95, Proposition 18.13] and contains an open dense geometrically integral subscheme. Thus $X$ is geometrically integral and it satisfies

$$\operatorname{Tor}^{\mathbb{Z}/2}(X \times_{k(t)} \overline{k(t)}, W \times_{k(t)} \overline{k(t)}) = 2$$

for the closed subscheme $W \coloneqq \{x_1 x_2 z_1 z_3 = 0\} \subset X$.

**Corollary 5.3.12.** *Let $f_1, f_2 \in k[x_1, x_2, x_3, z_1, z_2, z_3]$ be the polynomials as in* (5.3.13) *and let $K \coloneqq \overline{k(t)}$ be the algebraic closure of a purely transcendental field extension*

*$k(t)/k$ of transcendence degree 1. Then the following holds:*

*(i) The $K$-algebra $B := K[x_1, x_2, x_3, x_4, z_2, z_3]/(t - x_4(f_1 + x_4))$ is strongly $K$-rational;*

*(ii) $X := \operatorname{Spec} B[z_1]/(f_2)$ is an integral $K$-variety of dimension 5;*

*(iii) Set $W := \{x_1x_2z_1z_3 = 0\} \subset X$, then $\operatorname{Tor}^{\mathbb{Z}/2}(X, W) = 2$;*

*(iv) For any field extension $F/k$ and $q \in F$, there exists an $F$-algebra epimorphism*

$$F[x_1, x_2, x_3, x_4, z_1, z_2, z_3]_{\partial_{z_1} f_2}/(t - x_4(f_1 + x_4), f_2 + q) \twoheadrightarrow F.$$

*Proof.* The first statement follows directly from the isomorphism of $K$-algebras

$$B = K[x_1, x_2, x_3, x_4, z_2, z_3]/(t - x_4(x_3 - z_3^2 + x_4)) \simeq K[x_1, x_2, x_4, z_2, z_3]_{x_4}.$$

The statement (ii) and (iii) are proven in Example 5.3.11. To prove statement (iv), let $F/K$ be a field extension and let $q \in F$. Then the quotient by the ideal

$$\left(x_4 - 1, z_2 - 2, 1 - x_1 + x_2, z_1 - 1, z_3 - \sqrt{1-t}\right)$$

yields for $q \neq 0$ an $F$-algebra epimorphism

$$F[x_1, x_2, x_3, x_4, z_1, z_2, z_3]_{\partial_{z_1} f_2}/(t - x_4(f_1 + x_4), f_2 + q) \twoheadrightarrow F[x_1]/(x_1 + q) \simeq F.$$

For the $q = 0$, (5.3.14) implies the existence of such an $F$-algebra epimorphism. $\square$

## 5.4. Applications and main results

In this section we apply the machinery developed in Section 5.2 to the base examples in Section 5.3 in order to construct certain affine complete intersections. Taking the closure of these complete intersections in different compactification of affine space $\mathbb{A}^N$ enables us to prove the main results stated in the introduction.

### 5.4.1. Complete intersections

We start with the most straightforward compactification of $\mathbb{A}^N$, namely $\mathbb{P}^N$. In the following proposition, we construct affine complete intersections with given (relative) torsion order. Their projective closure can be controlled via the theory of Gröbner basis and graded monomial orderings, see Proposition 2.6.4.

**Proposition 5.4.1.** *Let $k$ be an uncountable and algebraically closed field and let $n, m \geq 2$ be two integers such that $m \in k^*$. For $s \geq 1$, consider an $s$-tuple $(d_1, \dots, d_s) \in \mathbb{Z}^s_{\geq 1}$ of positive integers satisfying $d_1 \geq n+m$.*

*Then for all integers $N$ and $M$ satisfying the inequalities*

$$s-1 \leq M \leq \sum_{i=1}^{s} d_i - n - m \quad \text{and} \quad 4 \leq N \leq n + 2^n - 1 + \sum_{j=0}^{n-1} \binom{n}{j} \left\lfloor \frac{j}{m} \right\rfloor, \tag{5.4.1}$$

*there exist polynomials $g_1, \dots, g_s \in k[x_1, \dots, x_N, y_1, \dots, y_M]$ of degree $\deg g_i = d_i$ such that the scheme*

$$X \coloneqq \operatorname{Spec} k[x_1, \dots, x_N, y_1, \dots, y_M]/(g_1, \dots, g_s)$$

*is an (integral) $k$-variety of dimension $N+M-s$ and satisfies $\operatorname{Tor}^{\mathbb{Z}/m}(X, W) = m$, where $W \coloneqq \{l \cdot \partial_{x_1} g_1 = 0\} \subset X$ for some non-zero polynomial $l \in k[x_1, \dots, x_N]$.*

*Moreover, if $s \leq M$ then the leading monomials of $g_1, \dots, g_s$ are relatively prime for some graded monomial ordering.*

*Proof.* Let $s \geq 1$ and let $(d_1, \dots, d_s) \in \mathbb{Z}^s_{\geq 1}$ be a tuple of positive integers such that $d_1 \geq n+m$. We can assume without loss of generality that each $d_i \geq 2$, as $d_1 \geq n+m = 4$ and for any $k$-algebra $A$ we have a $k$-algebra isomorphism $A[z]/(z) \simeq A$. Let $N, M, d$ be positive integers satisfying the inequalities in (5.4.1). Since $N$ satisfies the bounds in (5.4.1), Theorem 5.3.1 shows that for every integer $d' \geq n+m$ there exist non-zero polynomials

$$f \in k[x_1, \dots, x_N] \quad \text{and} \quad l \in k[x_1, \dots, x_N] \tag{5.4.2}$$

such that $f$ is irreducible of degree $d'$ and satisfies condition ($\star\star$) with $z = x_1$ as well as

$$\operatorname{Tor}^{\mathbb{Z}/m}(\operatorname{Spec} k[x_1, \dots, x_N]/(f), \operatorname{Spec} k[x_1, \dots, x_N]/(f, l \cdot \partial_{x_1} f)) = m. \tag{5.4.3}$$

In particular, the properties (C1)–(C4) in Theorem 5.2.12 are satisfied for $f$ and the integers $n = N-1$ and $r = 0$. We will use the existence of such $f$ several times throughout the proof.

If $s = 1$, then we set $d' \coloneqq d_1 - M$ and note that $d' \geq m+n$ by (5.4.1). Consider the irreducible polynomial $f \in k[x_1, \dots, x_N]$ of degree $d'$ from (5.4.2). Then, we define a polynomial $g_1 \in k[x_1, \dots, x_N, w_1, \dots, w_M]$ of degree $d_1$ by setting $g_1 \coloneqq f$, if $M = 0$, or by applying Theorem 5.2.12 (b) to $f$ and with $r = 0$, added degree $M$ and $M$ added variables and setting $g_1 \coloneqq \check{f}$, if $M > 0$. In both cases we know that the affine scheme

$$\operatorname{Spec} k[x_1, \dots, x_N, y_1, \dots, y_M]/(g_1)$$

is an integral $k$-variety of dimension $N + M - 1$ and satisfies $\operatorname{Tor}^{\mathbb{Z}/m}(X, W) = m$, where $W \coloneqq \{l \cdot \partial_{x_1} g_1 = 0\} \subset X$ for $l$ as in (5.4.2). This proves the case $s = 1$. Assume from now on $s \geq 2$. We distinguish three cases depending on $M$.

**Case a.** If $M \geq 2s - 2$, we set $d' \coloneqq \min\{d_1, d_1 + \dots + d_s - M\}$. Note that $n + m \leq d' \leq d_1$ by (5.4.1). Since each $d_i \geq 2$ by assumption, we have

$$2s - 2 \leq M + d' - d_1 \leq d_2 + \dots + d_s.$$

Thus there exist (not necessarily unique) integers $M_2, \dots, M_s \in \mathbb{Z}_{\geq 2}$ such that $2 \leq M_i \leq d_i$ and $M_2 + \dots + M_r = M + d' - d_1$. We set $M_1 \coloneqq d_1 - d'$. Let $f \in k[x_1, \dots, x_N]$ be the irreducible polynomial of degree $d'$ as in (5.4.2). By applying Theorem 5.2.12 (b) to $f$ with added degree $M_1$ and $M_1$ added variables if $M_1 > 0$, we can assume that there exists an irreducible polynomial $\check{f} \in k[x_1, \dots, x_N, y_1, \dots, y_{M_1}]$ of degree $d_1$, which satisfies the properties (C1)–(C4) in Theorem 5.2.12. Then we repeatedly apply Theorem 5.2.12 (a) to $\check{f}$ with added degrees $d_2, \dots, d_s$ and $M_2, \dots, M_s$ added variables in order to obtain polynomials

$$\begin{aligned} &g_1 \in k[x_1, \dots, x_N, y_1, \dots, y_M],\ g_2 \in k[y_{M_1+1}, \dots, y_{M_1+M_2}], \\ &g_3 \in k[y_{M_1+M_2+1},\ \dots, y_{M_1+M_2+M_3}], \dots,\ g_s \in k[y_{M-M_s+1}, \dots, y_M] \end{aligned} \tag{5.4.4}$$

of degree $\deg g_i = d_i$ such that the scheme

$$X \coloneqq \operatorname{Spec} k[x_1, \dots, x_N, y_1, \dots, y_M]/(g_1, \dots, g_s)$$

is an integral $k$-variety of dimension $N + M - s$ (see (C2)) and satisfies $\operatorname{Tor}^{\mathbb{Z}/m}(X, W) = m$, where $W \coloneqq \{l \cdot \partial_{x_1} g_1 = 0\} \subset X$ for some $l \in k[x_1, \dots, x_N]$ (see (C3)). In fact, $l$ can be choosen as in (5.4.2), see Remark 5.2.13. The claim about the leading monomials follows directly from (5.4.4) by choosing the graded lexicographical ordering, see Example 2.6.1. Note that this uses implicitely that the degree of $g_1$ does not change under the construction in Theorem 5.2.12 (a), see Remark 5.2.13.

**Case b.** Assume $s \leq M < 2s - 2$. Define the integer $s' \coloneqq M + 2 - s$ and note that

$2 \leq s' < s$ by the assumption on $M$. Similar to Case a, we apply Theorem 5.2.12 (a) repeatedly $s'-1$ times with added degrees $d_2, \ldots, d_{s'}$ and $2, 2, \ldots, 2$ added variables starting with the polynomial $f \in k[x_1, \ldots, x_N]$ of degree $d' \coloneqq d_1$ as in (5.4.2). Thus, we obtain polynomials

$$g_1 \in k[x_1, \ldots, x_N, y_1, \ldots, y_{2s'-2}],\ g_2 \in k[y_1, y_2],\ \ldots,\ g_{s'} \in k[y_{2s'-3}, y_{2s'-2}] \quad (5.4.5)$$

of degree $\deg g_i = d_i$ as well as $\deg_x g_1 = d_1$ such that

$$X' \coloneqq \operatorname{Spec} k[x_1, \ldots, x_N, y_1, \ldots, y_{2s'-2}]/(g_1, \ldots, g_{s'})$$

is an integral $k$-variety of dimension $N + s' - 2 = N + M - s$ (see (C2)) and satisfies $\operatorname{Tor}^{\mathbb{Z}/m}(X', W') = m$, where $W' \coloneqq \{l \cdot \partial_{x_1} g_1 = 0\} \subset X'$ for $l' \in k[x_1, \ldots, x_N]$ as in (5.4.2) (see (C3)). By Remark 5.2.13 (c), we can assume that

$$g_{s'} = a - y_{2s'-3}y_{2s'-2} + by_{2s'-3}^{d_{s'}} \in k[y_{2s'-3}, y_{2s'-2}]. \quad (5.4.6)$$

for some elements $a, b \in k^*$. We define additionally the polynomials

$$g_{s'+1} \coloneqq y_{2s'-1} + y_{2s'-2}^{d_{s'+1}},\ g_{s'+2} \coloneqq y_{2s'} + y_{2s'-1}^{d_{s'+2}},\ \ldots,\ g_s \coloneqq y_{s+s'-2} + y_{s+s'-3}^{d_s} \quad (5.4.7)$$

in $k[y_{2s-1}, \ldots, y_M]$ of degree $\deg g_i = d_i$. Then there is an obvious isomorphism of affine $k$-schemes

$$\begin{aligned} X &\coloneqq \operatorname{Spec} k[x_1, \ldots, x_N, y_1, \ldots, y_M]/(g_1, \ldots, g_s) \\ &\simeq \operatorname{Spec} k[x_1, \ldots, x_N, y_1, \ldots, y_{2s'-2}]/(g_1, \ldots, g_{s'}) = X', \end{aligned}$$

where $g_1, \ldots, g_{s'}$ are as in (5.4.5). In particular, $X$ is an affine $k$-variety of dimension $N + M - s$. The isomorphism sends $W' \subset X'$ to $W \coloneqq \{l \cdot \partial_{x_1} g_1 = 0\} \subset X$, which immediately implies that $\operatorname{Tor}^{\mathbb{Z}/m}(X, W) = \operatorname{Tor}^{\mathbb{Z}/m}(X', W') = m$. The explicit forms of $g_1, \ldots, g_s$ in (5.4.5), (5.4.6), and (5.4.7) imply that the leading monomials are relatively prime for the graded lexicographical ordering (Example 2.6.1).

**Case c.** If $M = s - 1$, then consider the polynomial $g_1 \coloneqq f \in k[x_1, \ldots, x_N]$ as in (5.4.2) and define the polynomials

$$g_2 \coloneqq y_1 + h_2(x_1, \ldots, x_N),\ g_3 \coloneqq y_2 + h_3(x_1, \ldots, x_N),\ \ldots,\ g_s \coloneqq y_M + h_s(x_1, \ldots, x_N)$$

in $k[x_1, \ldots, x_N, y_1, \ldots, y_M]$, where $h_2, \ldots, h_s \in k[x_1, \ldots, x_N]$ are arbitrary polynomials of degree $\deg h_i = d_i$. Then the claims in the proposition reduce in this case via the obvious isomorphism of affine $k$-schemes

$$\operatorname{Spec} k[x_1, \dots, x_N, y_1, \dots, y_M]/(g_1, \dots, g_s) \simeq \operatorname{Spec} k[x_1, \dots, x_N]/(f) =: X'$$

to the same statements about $X'$, which are known by Theorem 5.3.1. □

As a first consequence, we obtain a new lower bound on the torsion order of complete intersections in projective space. Recall that any (smooth) complete intersection of dimension at least 2 in projective space satisfies $h^{1,0} = 0$, see e.g. [SGA7, Exposé XI, Théorème 1.5]. In particular, its torsion order is infinite or it has trivial Chow group of zero-cycles ($\mathrm{CH}_0(-) \simeq \mathbb{Z}$), see Lemma 4.2.7.

**Theorem 5.4.2.** *Let $k$ be a field and let $n, m \geq 2$ be integers. Then the torsion order of a very general complete intersection of multidegree $(d_1, \dots, d_s)$ and dimension $D \geq 4$ is divisible by $m$, if $d_1 \geq n + m$ and the Fano index*

$$r := D + s + 1 - \sum_{i=1}^{s} d_i \leq 2^n + \sum_{j=0}^{n-1} \binom{n}{j} \left\lfloor \frac{j}{m} \right\rfloor - m. \tag{5.4.8}$$

*Proof.* Since the torsion order of any $k$-variety is divisible by the torsion order of its base change to any field extension $L/k$ and the notion of very general is stable under field extension by Lemma 2.4.3 (ii), we can assume without loss of generality that the field $k$ is algebraically closed and uncountable.

Let $(d_1, \dots, d_s) \in \mathbb{Z}^s_{\geq 1}$ and $D \geq 4$ be positive integers satisfying $d_1 \geq n + m$ and

$$D + s + 1 - \sum_{i=1}^{s} d_i \leq 2^n + \sum_{j=0}^{n-1} \binom{n}{j} \left\lfloor \frac{j}{m} \right\rfloor - m.$$

Then we easily find integers $N$ and $M$ with

$$s \leq M \leq \sum_{j=1}^{s} d_i - n - m \quad \text{and} \quad 4 \leq N \leq n + 2^n - 1 + \sum_{j=0}^{n-1} \binom{n}{j} \left\lfloor \frac{j}{m} \right\rfloor$$

such that $D = N + M - s$. By Proposition 5.4.1 with $N$ and $M$ as above, we know that there exist polynomials

$$f_1, \dots, f_s \in k[x_1, \dots, x_N, y_1, \dots, y_M]$$

of degree $\deg f_i = d_i$ such that the leading monomials of $f_1, \dots, f_s$ are relatively prime for some graded monomial ordering and the affine $k$-variety

$$X^\circ := \operatorname{Spec} k[x_1, \dots, x_N, y_1, \dots, y_M]/(f_1, \dots, f_s)$$

has dimension $D$ and satisfies $\operatorname{Tor}^{\mathbb{Z}/m}(X^\circ, W^\circ) = m$ for some closed $W^\circ \subset X^\circ$.

Let $X$ be the projective closure of $X^\circ$ in $\mathbb{P}^{N+M}$ and denote by $H_\infty$ the hyperplane $\mathbb{P}^{N+M} \setminus \mathbb{A}^{N+M}$. Then it follows immediately that $X$ is a (projective) $k$-variety of dimension $D$ satisfying

$$\operatorname{Tor}^{\mathbb{Z}/m}(X, W) = m, \tag{5.4.9}$$

where $W \subset X$ is the unique reduced closed subscheme satisfying $X \setminus W = X^\circ \setminus W^\circ$ in $X$. Moreover, Proposition 2.6.4 implies that

$$X = \operatorname{Proj} k[x_0, \dots, x_N, y_1, \dots, y_M]/(f_1^h, \dots, f_s^h),$$

where $f_i^h \in k[x_0, \dots, x_N, y_1, \dots, y_M]$ is the homogenization of the polynomial $f_i$, see Definition 2.6.3. Thus $X$ is an integral complete intersection of multi-degree $(d_1, \dots, d_s)$ in $\mathbb{P}^{N+M} = \mathbb{P}^{D+s}$ such that the $\mathbb{Z}/m$-torsion order relative to some non-empty closed subscheme is equal to $m$.

Let $X_{d_1,\dots,d_s}$ be a very general complete intersection of multidegree $(d_1, \dots, d_s)$ in $\mathbb{P}^{D+s}$ over $k$. We aim to show that $m$ divides $\operatorname{Tor}(X_{d_1,\dots,d_s})$ by using (5.4.9). By Lemma 2.4.3 (iii), up to a base change to an algebraically closed field extension of $k$ the variety $X_{d_1,\dots,d_s}$ degenerates to $X$. Since the torsion order is stable under base changes to algebraically closed field extensions, see Lemma 4.2.6, we can assume that $X_{d_1,\dots,d_s}$ degenerates to $X$. The intersection of $H_\infty$ and $D-1$ general hyperplane sections yields a closed subscheme $\mathcal{W}$ of the total space of the degeneration, which has relative dimension 0 and whose restriction to $X$ is contained in $W$. The latter uses that $H_\infty \cap X \subset W$ by construction. Thus, we conclude from Lemma 4.2.9 applied to the total space of the degeneration with closed subscheme $\mathcal{W}$ that

$$\operatorname{Tor}^{\mathbb{Z}/m}(X_{d_1,\dots,d_s}, W_{d_1,\dots,d_s}) = \operatorname{Tor}^{\mathbb{Z}/m}(X, W) \overset{(5.4.9)}{=} m,$$

where $W_{d_1,\dots,d_s} := \mathcal{W} \cap X_{d_1,\dots,d_s}$ is a non-empty zero-dimensional closed subscheme of $X_{d_1,\dots,d_s}$. In particular, we find that $m$ divides the torsion order $\operatorname{Tor}(X_{d_1,\dots,d_s})$ by Lemma 4.2.6, as the torsion order of $X_{d_1,\dots,d_s}$ is infinite or $\operatorname{CH}_0(X_{d_1,\dots,d_s}) \simeq \mathbb{Z}$. $\square$

*Proof of Theorem 5.1.1:* Let $r, m \in \mathbb{Z}_{\geq 1}$ be positive integers. If $m = 1$, then the statement of the theorem is trivial, so we can assume that $m \geq 2$. We aim to deduce the theorem from Theorem 5.4.2. Up to reordering $(d_1, \dots, d_s) \in \mathbb{Z}^s_{\geq 1}$, we can assume that $d_1 \geq \log_2(r+m) + m$. We set $n \coloneqq d_1 - m$ and note that

$$n \geq \lceil \log_2(r+m) \rceil \geq \lceil \log_2(3) \rceil = 2.$$

Note also that the dimension of a complete intersection $X$ of multidegree $(d_1, \dots, d_s)$ with positive Fano index $r > 0$ is at least

$$\dim X \geq \sum_{i=1}^{s} d_i - s \geq d_1 - 1 \geq \lceil \log_2(1+m) \rceil + m - 1 \geq 3.$$

We observe that $\dim X = 3$ if and only if $s = 1$, $m = 2$ and $d_1 = 4$. Note that the statement for quartic threefolds has been proven in [CTP16], see also [Sch21a, Theorem 1.1]. Thus we can assume that $\dim X \geq 4$. Since

$$r \leq 2^n - m \leq 2^n + \sum_{j=0}^{n-1} \binom{n}{j} \left\lfloor \frac{j}{m} \right\rfloor - m,$$

Theorem 5.1.1 follows directly from Theorem 5.4.2. □

### 5.4.2. Complete intersections of small Fano index

Starting from the examples in Section 5.3.2, we obtain additional affine complete intersection of small Fano index.

**Proposition 5.4.3.** *Let $k$ be an uncountable algebraically closed field of characteristic different from 2. Let $(d_1, \dots, d_s) \in \mathbb{Z}^s_{\geq 2}$ be an $s$-tuple for some integer $s \geq 1$. Then for all integers $M$ satisfying*

$$s \leq M \leq d_1 + \dots + d_s - 3,$$

*there exist polynomials $f_1, \dots, f_s \in k[x_1, \dots, x_{4+M}]$ of degree $\deg f_i = d_i$ whose leading monomials are relative prime for some graded monomial ordering such that*

$$X \coloneqq \operatorname{Spec} k[x_1, \dots, x_{4+M}]/(f_1, \dots, f_s)$$

*is a (integral) $k$-variety of dimension $4 + M - s$ and satisfies $\operatorname{Tor}^{\mathbb{Z}/2}(X, W) = 2$ for some non-empty closed subscheme $W \subset X$.*

The construction of $f_1, \dots, f_s$ is very similar to the construction in Proposition 5.4.1 replacing the base example from Section 5.3.1 with the examples from

Section 5.3.2. We provide some details for the convenience of the reader.

*Proof.* Up to reordering the $d_i$'s, we can assume that $d_1 \geq d_2 \geq \cdots \geq d_s$. If $d_1 \geq 4$, then the proposition is a special case of Proposition 5.4.1 with $n = m = 2$ and $N = 4$, except if $M = d_1 + \cdots + d_s - 3$, then take $N = 5$. So we can assume from now on that each $d_i \in \{2, 3\}$. We distinguish three cases.

**Case a.** Assume $d_1 = \cdots = d_s = 2$ and $s \geq 3$. Define $s_0 \coloneqq 2s - M$ and note that $3 \leq s_0 \leq s$. Let $q_1, q_2, q_3 \in k[x_1, \ldots, x_7]$ be the polynomials as in (5.3.12) and define the polynomial

$$q_i \coloneqq x_{4+i} + x_{3+i}^2 \in k[x_1, \ldots, x_{4+s_0}].$$

for $i \in \{4, \ldots, s_0\}$. Then we have an isomorphism of $k$-algebras

$$k[x_1, x_3, \ldots, x_{4+s_0}]/(q_1, q_3, q_4 \ldots, q_{s_0}) \simeq k[x_1, x_3, \ldots, x_7]/(q_1, q_3).$$

Thus, it follows from (5.3.10) that the assumption (C1)–(C4) of Theorem 5.2.12 are satisfied for $q_1, q_3, q_4, \ldots, q_{s_0}$ and $f = q_2$ with $z = x_2$. Applying Theorem 5.2.12 (a) with added degree 2 and 2 added variables $(s-s_0)$ times yields quadratic polynomials

$$\begin{aligned} &f_1 \in k[x_1, \ldots, x_{4+2s-s_0}], \quad f_2 \coloneqq q_1 \in k[x_1, \ldots, x_{4+s_0}], \\ &f_i \coloneqq q_i \in k[x_1, \ldots, x_{s_0}], \quad f_{s_0+j} \in k[x_{4+s_0+2j-1}, x_{4+s_0+2j}] \end{aligned}$$

for $i \in \{3, \ldots, s_0\}$ and $j \in \{1, \ldots, s - s_0\}$ such that the affine scheme

$$X \coloneqq \operatorname{Spec} k[x_1, \ldots, x_{4+M}]/(f_1, \ldots, f_s)$$

is an integral $k$-variety of dimension $4 + M - s$ and $\operatorname{Tor}^{\mathbb{Z}/2}(X, W) = 2$ for some non-empty closed subscheme $W \subset X$.

Note that the leading monomials of the polynomials $q_1, \ldots, q_{s_0}$ with respect to the graded monomial ordering given by $x_2 > x_3 > \cdots > x_{4+s_0} > x_1$ are

$$\mathrm{LM}(q_1) = x_3^2, \quad \mathrm{LM}(q_2) = x_2^2, \quad \mathrm{LM}(q_3) = x_5^2, \quad \mathrm{LM}(q_i) = x_{3+i}^2$$

for $i \in \{4, \ldots, s_0\}$; in particular the leading monomials of $q_1, \ldots, q_{s_0}$ are relatively prime. It follows from Theorem 5.2.12 that the leading monomials of the polynomials $f_1, \ldots, f_s$ are relatively prime for the graded monomial ordering given by

$$x_2 > x_3 > \cdots > x_{4+M} > x_1,$$

see also Remark 5.2.13 (c). Thus, we can assume now $d_1 = 3$.

**Case b.** Assume $d_1 = 3$ and $M \neq 3s - 3$. We aim to apply a similar argument as in Proposition 5.4.1, which we spell out for the convenience of the reader.

Let $f \in k[x_1, x_2, z_1, z_2, z_3]$ be the polynomial of degree 4 as in (5.3.9). It is shown in Example 5.3.6 and Lemma 5.3.7 that the polynomial $f$ satisfies the assumption (C1)–(C4) of Theorem 5.2.12 for $r = 0$. Define

$$s_1 \coloneqq \max\{e \in \{0, 1, \ldots, s-2\} : M - s - e \geq 0\}$$

and set $M_0 \coloneqq M - s + s_1 \in \mathbb{Z}$. Note that $2s_1 \leq M_0 \leq d_2 + d_3 + \cdots + d_{s_1+1}$. Then there exists integers $M_2, \ldots, M_{s_1+1} \in \mathbb{Z}_{\geq 2}$ such that $M_2 + \cdots + M_{s_1+1} = M_0$. By applying Theorem 5.2.12 (a) $s_1$-times to the polynomial $f$ with added degree $d_2, \ldots, d_{s_1+1}$ and $M_2, \ldots, M_{s_1+1}$ added variables, we find polynomials

$$\begin{aligned} &\tilde{f} \in k[x_1, x_2, z_1, z_2, z_3, y_1, \ldots, y_{M_0}],\ f_2 \in k[y_1, \ldots, y_{M_2}], \\ &f_3 \in k[y_{M_2+1}, \ldots, y_{M_2+M_3}],\ \ldots\ ,\ f_{s_1+1} \in k[y_{M_0 - M_{s_1+1}}, \ldots, y_{M_{s_1+1}}] \end{aligned} \tag{5.4.10}$$

of degree $\deg \tilde{f} = 4$ and $\deg f_i = d_i$ for $i = 2, \ldots, s_1 + 1$ such that

$$X \coloneqq \operatorname{Spec} k[x_1, x_2, z_1, z_2, z_3, y_1, \ldots, y_{M_0}]/(\tilde{f}, f_2, f_3, \ldots, f_{s_1+1})$$

is a $k$-variety of dimension $M_0 + 4 - s_1$ satisfying $\mathrm{Tor}^{\mathbb{Z}/2}(X, W) = 2$ for some non-empty closed subscheme $W \subset X$. Moreover, the polynomial $\tilde{f}$ is of the form $\tilde{f} = f + h$ for some linear polynomial $h \in k[y_1, \ldots, y_{M_0}]$, see Remark 5.2.13 (b).

For each $j \in \{s_1 + 2, \ldots, s - 1\}$ define the polynomial

$$f_j \coloneqq \begin{cases} z_2^{d_j} + z_4 & \text{if } j = s_1 + 2, \\ z_{j+1-s_1}^{d_j} + z_{j+2-s_1} & \text{otherwise.} \end{cases} \tag{5.4.11}$$

in $k[z_2, z_4, z_5, \ldots, z_{s+1-s_1}]$. Then we have

$$\operatorname{Spec} k[x_1, x_2, z_1, \ldots, z_{s+1-s_1}, y_1, \ldots, y_{M_0}]/(\tilde{f}, f_2, f_3, \ldots, f_{s-1}) \simeq X.$$

Consider the following polynomials

$$f_1 \coloneqq x_1 z_1^2 + x_2 z_2^2 + x_1 x_3^{3-d_s} + (1 + x_1^2 + x_2^2 - 2x_1 - 2x_2 - 2x_1 x_2) + h, \quad f_s \coloneqq x_3 - x_2 z_3^{d_s - 1},$$

in $k[x_1, x_2, x_3, z_1, \ldots, z_{s+1-s_1}, y_1, \ldots, y_{M_0}]$, where $h \in k[y_1, \ldots, y_{M_0}]$ is the linear polynomial $h = \tilde{f} - f$ for $f$ as in (5.3.9), see Remark 5.2.13 (b). Then the above isomorphism yields

$$X \simeq \operatorname{Spec} k[x_1, x_2, x_3, z_1, \dots, z_{s+1-s_1}, y_1, \dots, y_{M_0}]/(f_1, \dots, f_s),$$

Note that $3 + s + 1 - s_1 + M_0 = 4 + M$. The leading monomials of the polynomials $f_1, \dots, f_s$ are relative prime for the graded monomial ordering

$$x_1 > z_1 > x_2 > x_3 > z_2 > \cdots > z_{s+1-s_1} > y_1 > y_2 > \cdots > y_{M_0},$$

which concludes this case.

**Case c.** Assume $d_1 = 3$ and $M = 3s - 3$, in particular $d_1 = \cdots = d_s = 3$. By Corollary 5.3.12, there exist cubic polynomials $c_1, c_2 \in k[x_1, x_2, \dots, x_7]$ such that the assumptions (C1)–(C4) of Theorem 5.2.12 are satisfied. By applying Theorem 5.2.12 (a) $(s-2)$-times with added degree 3 and 3 added variable each time, we obtain

$$f_1, \dots, f_s \in k[x_1, \dots, x_{3s+1}]$$

with degree $\deg f_i = 3$ such that the affine scheme

$$X \coloneqq \operatorname{Spec} k[x_1, \dots, x_{3s+1}]/(f_1, \dots, f_s)$$

is an integral $k$-variety of dimension $2s + 1$ and satisfy $\operatorname{Tor}^{\mathbb{Z}/2}(X, W) = 2$ for some non-empty closed subscheme $W \subset X$, which completes the third and final case. $\square$

**Theorem 5.4.4.** *Let $k$ be a field of characteristic different from 2. Then the torsion order of a very general complete intersection $X \subset \mathbb{P}^N$ of multidegree $(d_1, \dots, d_s) \in \mathbb{Z}^s_{\geq 2}$ is divisible by 2 if $N \geq 4 + s$ and the Fano index $r \coloneqq N + 1 - d_1 - \cdots - d_s \leq 2$.*

**Remark 5.4.5.** Theorem 5.4.4 covers the complete intersection of $s_1$ cubic hypersurface and $s_2$ quadrics in $\mathbb{P}^{3s_1+2s_2+1}$, in particular the complete intersection of $s$ quadrics in $\mathbb{P}^{2s+1}$. These cases are not contained in Theorem 5.4.2. The special case of a $(2, 3)$-fourfold, originally due to Skauli [Ska23], was recently reproved by Fiammengo–Lüders [FL25] using the methods of [PS23, LS25]. The input from [HPT18a] here is replaced with the stable irrationality result of cubic threefolds due to Engel–de Gaay Fortman–Schreieder [EGFS25], which builds on [Voi17].

*Proof.* We prove the theorem by essentially the same argument as in Theorem 5.4.2. Since the torsion order and the notion of very general behave well under field extensions, we can assume that $k$ is algebraically closed and uncountable. By Proposition 5.4.3, there exists an affine complete intersection $X^\circ$ of multidegree $(d_1,\dots,d_s)$ in $\mathbb{A}^N_k$ such that $\operatorname{Tor}^{\mathbb{Z}/2}(X^\circ,W^\circ)=2$ for some non-empty closed subscheme $W^\circ\subset X^\circ$. Moreover, the leading monomial of a set of defining equations of $X^\circ$ are relatively prime for some graded monomial ordering. Let $X$ be the closure of $X^\circ$ in $\mathbb{P}^N$ and denote by $H_\infty$ the hyperplane $\mathbb{P}^N\setminus\mathbb{A}^N$. Proposition 2.6.4 shows that $X$ is a complete intersection in $\mathbb{P}^N$ of multidegree $(d_1,\dots,d_s)$ such that

$$\operatorname{Tor}^{\mathbb{Z}/2}(X,W)=\operatorname{Tor}^{\mathbb{Z}/2}(X^\circ,W^\circ)=2$$

for the non-empty closed subset $W\subset X$ satisfying $X\setminus W=X^\circ\setminus W^\circ\subset X$.

Let $X_{d_1,\dots,d_s}$ be a very general complete intersection in $\mathbb{P}^N$ of multidegree $(d_1,\dots,d_s)$. Since the base change to an algebraically closed field extension does not affect the torsion order by Lemma 4.2.6, we can assume that $X_{d_1,\dots,d_s}$ degenerates to $X$ by Lemma 2.4.3 (iii). The intersection of $X_{d_1,\dots,d_s}$ with $H_\infty$ and $\dim X-1$ general hyperplanes gives a non-empty zero-dimensional closed subscheme $W_{d_1,\dots,d_s}\subset X_{d_1,\dots,d_s}$ specializing to a closed subscheme of $W$. Thus, Lemma 4.2.9 implies that

$$2\mid\operatorname{Tor}^{\mathbb{Z}/2}(X_{d_1,\dots,d_s},W_{d_1,\dots,d_s}).$$

In particular, we find that 2 divides $\operatorname{Tor}(X_{d_1,\dots,d_s})$ by Lemma 4.2.6, as $\operatorname{Tor}(X_{d_1,\dots,d_s})=\infty$ or $\mathrm{CH}_0(X_{d_1,\dots,d_s})\simeq\mathbb{Z}$ by Lemma 4.2.7. □

*Proof of Theorem 5.1.2:* The case $r\le 2$ is exactly Theorem 5.4.4. We turn to the other case. Let $d_i\ge 4$ be a integers and set $n\coloneqq d_i-2\ge 2$. A small computation shows that

$$2^n+\sum_{j=0}^{n-1}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor-2=2^n+(n-1)2^{n-2}-\left\lfloor\frac{n}{2}\right\rfloor-2=(n+3)2^{n-2}-\left\lfloor\frac{n+4}{2}\right\rfloor,$$

see Lemma 4.6.4. Thus, Theorem 5.4.2 with $m=2$ implies the theorem. □

### 5.4.3. Hypersurfaces in products of projective spaces

We aim to compactify $\mathbb{A}^N$ to a product of projective spaces. This leads to new bounds on the torsion order of hypersurfaces in products of projective spaces and subsequently also statements about the irrationality of such hypersurfaces.

**Proposition 5.4.6.** *Let $k$ be an uncountable and algebraically closed field and let $n, m \geq 2$ and $s \geq 0$ be integers such that $m \in k^*$. Then for any $(s+1)$-tuples*

$$(M_0, M_1, \ldots, M_s),\ (d_0, d_1, \ldots, d_s) \in \mathbb{Z}_{\geq 1}^{s+1}$$

*satisfying $d_0 \geq n+m$, $d_i \geq M_i + 1$ for all $i \in \{1, \ldots, s\}$, and*

$$4 \leq M_0 \leq n + 2^n - 1 + \sum_{j=0}^{n-1} \binom{n}{j} \left\lfloor \frac{j}{m} \right\rfloor, \tag{5.4.12}$$

*there exists an irreducible polynomial $f \in k[y_{0,1}, \ldots, y_{0,M_0}, y_{1,1}, \ldots, y_{1,M_1}, y_{2,1}, \ldots, y_{s,M_s}]$ of multidegree $(d_0, d_1, \ldots, d_s)$ such that*

$$\mathrm{Tor}^{\mathbb{Z}/m}\left(\operatorname{Spec} k[y_{0,1}, \ldots, y_{s,M_s}]/(f), \operatorname{Spec} k[y_{0,1}, \ldots, y_{s,M_s}]/(f, \partial_{y_{0,M_0}} f)\right) = m$$

*Proof.* Fix (non-negative) integers $n$, $m$, and $s$ as in the statement of the proposition. Let $M_0, M_1, \ldots, M_s, d_0, d_1, \ldots, d_s$ be positive integers satisfying $d_0 \geq n+m$, $d_i \geq M_i + 1$ for each $i \in \{1, \ldots, s\}$, and (5.4.12).

Since $d_0 \geq n+m$ and $M_0$ satisfies the bounds in (5.4.12), Theorem 5.3.1 shows that there exists an irreducible polynomial $f \in k[y_{0,1}, \ldots, y_{0,M_0}]$ of degree $d_0$ satisfying

$$\mathrm{Tor}^{\mathbb{Z}/m}\left(\operatorname{Spec} k[y_{0,1}, \ldots, y_{0,M_0}]/(f), \operatorname{Spec} k[y_{0,1}, \ldots, y_{0,M_0}]/(f, \partial_{y_{0,1}} f)\right) = m$$

and condition ($\star\star$) with $z = y_{0,1}$. Hence, the proposition holds for $s = 0$ and we can assume now that $s \geq 1$. Note that the conditions (C1)–(C4) in Theorem 5.2.12 are satisfied for $f$ and the integers $n = M_0 - 1$ and $r = 0$.

We repeatedly apply Theorem 5.2.12 (b) to $f$ with added degrees $d_1 - 1, \ldots, d_s - 1$ and $M_1, \ldots, M_s$ added variables and obtain an irreducible polynomial

$$\check{f} \in k[y_{0,1}, \ldots, y_{0,M_0}, y_{1,1}, \ldots, y_{1,M_1}, y_{2,1}, \ldots, y_{s,M_s}]$$

of multidegree $(d_0, d_1, \ldots, d_s)$ such that

$$\mathrm{Tor}^{\mathbb{Z}/m}\left(\operatorname{Spec} k[y_{0,1}, \ldots, y_{s,M_s}]/(\check{f}), \operatorname{Spec} k[y_{0,1}, \ldots, y_{s,M_s}]/(\check{f}, \partial_{y_{0,M_0}} \check{f})\right) = m,$$

which finishes the proof of the proposition. □

We consider the closure of the above affine examples in a product of projective spaces. Similar arguments as in Theorem 5.4.2 show then the following result.

**Theorem 5.4.7.** *Let $k$ be a field and let $n, m \geq 2$ be integers. Then the torsion order of a very general hypersurface in $\mathbb{P}^{M_0} \times \cdots \times \mathbb{P}^{M_s}$ of multidegree*

$$(d_0, d_1, \ldots, d_s) \geq (n+m, M_1+1, \ldots, M_s+1)$$

*is divisible by $m$, if*

$$4 \leq M_0 \leq n + 2^n - 1 + \sum_{l=0}^{n-1} \binom{n}{l} \left\lfloor \frac{l}{m} \right\rfloor .$$

*Proof.* Since the notion of very general is stable under field extension by Lemma 2.4.3 (ii) and the torsion order of a $k$-variety is divisible by the torsion order of its base change to a field extension, we can assume without loss of generality that the field $k$ is algebraically closed and uncountable.

Let $M_0, \ldots, M_s \in \mathbb{Z}_{\geq 1}$ be positive integers such that

$$4 \leq M_0 \leq n + 2^n - 1 + \sum_{l=0}^{n-1} \binom{n}{l} \left\lfloor \frac{l}{m} \right\rfloor .$$

For any $(s+1)$-tuple $(d_0, \ldots, d_s) \in \mathbb{Z}_{\geq 1}^{s+1}$ with $d_0 \geq n+m$ and $d_i \geq M_i + 1$ for $i \in \{1, \ldots, s\}$, Proposition 5.4.6 shows that there exists an irreducible polynomial

$$f^\circ \in k[y_{0,1}, \ldots, y_{0,M_0}, y_{1,1}, \ldots, y_{1,M_1}, y_{2,1}, \ldots, y_{s,M_s}]$$

of multidegree $(d_0, d_1, \ldots, d_s)$ such that the $\mathbb{Z}/m$-torsion order of the integral $k$-variety

$$X^\circ := \{f^\circ = 0\} \subset \mathbb{A}^{M_0} \times \cdots \times \mathbb{A}^{M_s}$$

relative to some closed subscheme $W^\circ \subset X^\circ$ is equal to $m$.

Let $Y := \mathbb{P}^{M_0} \times \cdots \times \mathbb{P}^{M_s}$ be the product of projective space and view $\mathbb{A}^{M_0+\cdots+M_s} \subset Y$ as a standard open subscheme; denote its complement by $H_\infty$. Let $X \subset Y$ be the closure of the locally closed subscheme $X^\circ$ inside $Y$ and let

$$f \in k[y_{0,0}, \ldots, y_{0,M_0}, y_{1,0}, \ldots, y_{1,M_1}, y_{2,0}, \ldots, y_{s,M_s}]$$

be the (multi-)homogenization of $f^\circ$. Then it follows immediately that

$$X = \{f = 0\} \subset Y = \mathbb{P}^{M_0} \times \cdots \times \mathbb{P}^{M_s},$$

is an integral hypersurface of multidegree $(d_0, d_1, \dots, d_s)$ satisfying $\mathrm{Tor}^{\mathbb{Z}/m}(X, W) = m$ for some closed non-empty subscheme $W \subset X$, in fact $W$ is the unique closed und reduced subscheme such that $X \setminus W = X^\circ \setminus W^\circ \subset X$.

Up to a base change to an algebraically closed field extension, which does not a effect the torsion order (Lemma 4.2.6), we can assume that a very general hypersurface $X_{d_0,\dots,d_s}$ of multidegree $(d_0, d_1, \dots, d_s)$ in $\mathbb{P}^{M_0} \times \cdots \times \mathbb{P}^{M_s}$ degenerates to $X$, see Lemma 2.4.3 (iii).

Consider the total space $\mathcal{X}$ of the degeneration, which is a hypersurface in the product of projective spaces $\mathbb{P}^{M_0} \times \cdots \times \mathbb{P}^{M_s}$ over some discrete valuation ring. The intersection $\mathcal{W}$ of $\mathcal{X}$ with $H_\infty$ and $M_0 + \cdots + M_s - 2$ general hyperplane sections gives a closed subscheme of relative dimension 0, whose intersection with $X$ is contained in $W$, as $X \cap H_\infty \subset W$ by construction.

Applying Lemma 4.2.9 we find that

$$\mathrm{Tor}^{\mathbb{Z}/m}(X_{d_0,\dots,d_s}, \mathcal{W} \cap X_{d_0,\dots,d_s}) = \mathrm{Tor}^{\mathbb{Z}/m}(X, W) = m, \tag{5.4.13}$$

where $\mathcal{W} \cap X_{d_0,\dots,d_s} \subset X_{d_0,\dots,d_s}$ is a non-empty zero-dimensional closed subscheme.

A simple computation using the Künneth formula shows that $H^{1,0}(X_{d_0,\dots,d_s}) = 0$, which uses that $M_0 > 1$. Thus Lemma 4.2.7 shows that $\mathrm{Tor}(X_{d_0,\dots,d_s}) = \infty$ or $\mathrm{CH}_0(X_{d_0,\dots,d_s}) = \mathbb{Z}$. In the latter case, Lemma 4.2.6 and (5.4.13) show that the $\mathrm{Tor}(X_{d_0,\dots,d_s})$ is divisible by $m$. □

*Proof of Theorem 5.1.3:* Let the assumptions be as in the statement of Theorem 5.1.3. We set $n := \log_2(M_1)$ and note that $n \geq \log_2(4) = 2$. Moreover,

$$M_1 = 2^n \leq 2^n + n - 1 + \sum_{j=0}^{n-1} \binom{n}{j} \left\lfloor \frac{j}{m} \right\rfloor.$$

Thus the statement follows directly from Theorem 5.4.7. □

### 5.4.4. Hypersurfaces in Grassmannians

To illustrate the flexibility of the method, we consider also hypersurfaces in Grassmannians over $\mathbb{C}$. Recall briefly some standard facts of Grassmannians, see e.g. [EH16, Section 3.2]. Let $k$ be a field and $l, n \in \mathbb{Z}_{\geq 1}$ be positive integers such that $l \leq n$. We denote the Grassmannian variety, whose closed points parametrize $l$-dimensional vector subspaces of $k^n$, by $\mathrm{Gr}(l, n)$. Throughout this section, we fix a Plücker embedding

$$\iota_{Pl}\colon \operatorname{Gr}(l,n) \hookrightarrow \mathbb{P}_k^{\binom{n}{l}-1} =: \mathbb{P},$$

which is a closed embedding. Let $U \subset \mathbb{P}$ be a standard open affine chart of the projective space $\mathbb{P}$. Then it follows from the construction of the Grassmannian that the scheme-theoretic intersection $V := U \cap \operatorname{Gr}(l,n) \subset \mathbb{P}$ is isomorphic to $\mathbb{A}^{l(n-l)}$. In fact, $V$ is the intersection of $\dim U - \dim V$ hyperplanes in $U$ see e.g. [EH16, Proposition 3.2].

**Lemma 5.4.8.** *Let the notation be as above. Then for any integral affine hypersurface $X^\circ \subset V \simeq \mathbb{A}^{l(n-l)}$ of degree $d$, there exists an integral affine hypersurface $Y^\circ \subset U \simeq \mathbb{A}^{\binom{n}{l}-1}$ of degree $d$ such that the scheme-theoretic closure $X$ of $X^\circ$ in $\operatorname{Gr}(l,n)$ is isomorphic to*

$$X \simeq Y \times_{\mathbb{P}} \operatorname{Gr}(l,n),$$

*where $Y \subset \mathbb{P}$ denotes the scheme-theoretic closure of $Y^\circ$ in $\mathbb{P} = \mathbb{P}^{\binom{n}{l}-1}$.*

*Proof.* Since $V$ is the intersection of hyperplanes in $U$, we can write the global sections of $U$ as $k[U] = k[V][y_1, \ldots, y_{\dim U - \dim V}]$. In particular, there exist morphisms of $k$-varieties

$$\iota_V : V \hookrightarrow U \quad \text{and} \quad p_V : U \longrightarrow V$$

such that $p_V \circ \iota_V = \mathrm{id}_V$. We claim that the fibre product $Y^\circ = U \times_V X^\circ$ satisfies the property claimed in the lemma. Note that $Y^\circ$ is an integral affine hypersurface of degree $d$, as $k[U] = k[V][y_1, \ldots, y_{\dim U - \dim V}]$. Consider the morphisms

$$f\colon Y^\circ \xrightarrow{p_V} X^\circ \hookrightarrow \operatorname{Gr}(l,n) \quad \text{and} \quad g\colon Y^\circ \hookrightarrow \mathbb{P} = \mathbb{P}^{\binom{n}{l}-1},$$

where the unspecificied arrows are the natural inclusions. Then it follows from the universal property of the fibre product that the scheme-theoretic image $\operatorname{Im} f$ of $f$ is isomorphic to the fibre product

$$\operatorname{Im} f \simeq \operatorname{Im} g \times_{\mathbb{P}} \operatorname{Gr}(l,n).$$

Note that $Y = \operatorname{Im} g$. Thus it suffices to show that $\operatorname{Im} f$ is equal to the scheme-theoretic closure $X$ of $X^\circ$ in $\operatorname{Gr}(l,n)$.

The equality of morphisms $p_V \circ \iota_V = \mathrm{id}_V$ implies that as sets $f(Y^\circ)$ is equal to the set-theoretic image of $X^\circ$ in $\operatorname{Gr}(l,n)$ via the natural inclusion. Since $Y^\circ$ and $X^\circ$ are reduced schemes, we find that $\operatorname{Im} f$ is equal to the scheme-theoretic closure $X$ of $X^\circ$ in $\operatorname{Gr}(l,n)$, see e.g. [GW20, Remark 10.32]. □

The lemma together with the examples of affine hypersurfaces in Theorem 5.3.1 shows the following theorem about the torsion order of very general hypersurfaces in Grassmannians.

**Theorem 5.4.9.** *Let $n', m \geq 2$ and $l, n \geq 1$ be positive integers such that $l \leq n$ and set $N := \binom{n}{l} - 1$. Fix a Plücker embedding $\mathrm{Gr}(l,n) \hookrightarrow \mathbb{P}^N$ of the Grassmannian $\mathrm{Gr}(l,n)$ over $\mathbb{C}$. Then the intersection of $\mathrm{Gr}(l,n)$ with a very general hypersurface of degree $d \geq n' + m$ in $\mathbb{P}^N_{\mathbb{C}}$ has torsion order divisible by $m$, if*

$$4 \leq \dim \mathrm{Gr}(l,n) = l(n-l) \leq 2^{n'} - 1 + \sum_{j=0}^{n'-1} \binom{n'}{j} \left\lfloor \frac{j}{m} \right\rfloor + d - m. \qquad (5.4.14)$$

*Proof.* Let $n', m, l, n$ be integers as in the theorem and write $N := \binom{n}{l} - 1$. Since $l(n-l)$ satisfies the bound (5.4.14), there exists by Proposition 5.4.1 an affine hypersurface

$$X^\circ \subset \mathbb{A}^{l(n-l)}$$

of degree $d \geq n' + m$ such that $\mathrm{Tor}^{\mathbb{Z}/m}(X^\circ, W^\circ) = m$ for some closed subscheme $W^\circ \subset X^\circ$.

Let $X \subset \mathrm{Gr}(l,n)$ and $Y \subset \mathbb{P}^N$ be the $k$-varieties constructed in Lemma 5.4.8; let $H_\infty := \mathbb{P}^N \setminus U$ be complement of the standard affine open $U = \mathbb{A}^N$ in $\mathbb{P}^N$. We denote by $W \subset X$ the unique reduced closed subscheme of $X$, whose complement is equal to $X^\circ \setminus W^\circ$ in $X$.

Let $Y_d$ be a very general hypersurface of degree $d$ in projective space $\mathbb{P}^N$ over $k$. Up to replacing $\mathbb{C}$ by an algebraically closed field extension, we can assume by Lemma 2.4.3 (iii) that $Y_d$ degenerates to $Y$. Let $X_d$ be the intersection of $Y_d$ with $\mathrm{Gr}(l,n)$, which is embedded in $\mathbb{P}^N$ via the fixed Plücker embedding. Then $X_d$ degenerates to $X$. The intersection of $X_d$ with $H_\infty$ and $\dim X_d - 1$ general hyperplanes is a non-empty closed zero-dimensional subscheme $W_d$ of $X_d$, which specializes to a closed subscheme of $W \subset X$.

Thus Lemma 4.2.9 implies that $\mathrm{Tor}^{\mathbb{Z}/m}(X_d, W_d) = m$. We aim to deduce that $m$ divides $\mathrm{Tor}(X_d)$, which could be $\infty$. By Lemma 4.2.7 and Lemma 4.2.6, it suffices to prove that $H^0(X_d, \Omega^1_{X_d}) = 0$. Consider the long exact sequence associtated to the conormal short exact sequence of the (Cartier) divisor $X_d$ in $\mathrm{Gr}(l,n)$

$$\cdots \longrightarrow H^0(\mathrm{Gr}(l,n), \Omega^1_{\mathrm{Gr}(l,n)}) \longrightarrow H^0(X_d, \Omega^1_{X_d}) \longrightarrow H^1(X_d, O_{X_d}(-d)) \longrightarrow \dots$$

It is well-known that the integral cohomology ring of a Grassmannian is generated by the Schubert classes and thus only non-trivial in even degree. In particular the first term in the above exact sequence vanishes. The vanishing of the last term follows directly from Kodaira's vanishing theorem applied to ample line bundles of the form $\mathcal{O}_{\mathrm{Gr}(l,n)}(j)$ on $\mathrm{Gr}(l,n)$ for $j > 0$. Thus we have $H^0(X_d, \Omega^1_{X_d}) = 0$ by exactness, which implies that $m$ divides $\mathrm{Tor}(X)$ by the above argument. □

**Remark 5.4.10.** The assumption on the base field being $\mathbb{C}$ is mostly for simplicity and the argument also works over arbitrary fields up to the computation of $h^{1,0}(X_d) = 0$.

*Proof of Theorem 5.1.4:* The statement follows from Theorem 5.4.9 together with the equality

$$\sum_{j=0}^{n'-1} \binom{n'}{j} \left\lfloor \frac{j}{2} \right\rfloor = (n'-1)2^{n'-2} - \left\lfloor \frac{n'}{2} \right\rfloor,$$

see Lemma 4.6.4. □

# 6. Specialization of algebraic equivalence

## 6.1. Introduction

We observe in this small note that algebraic equivalence specializes in strictly semi-stable families (see Definition 2.3.2) via a slightly refined version of Fulton's specialization map, which is constructed as follows:

Let $R$ be a discrete valuation ring with (perfect) residue field $k$ and fraction field $K$. Let $\mathcal{X}$ be a regular proper flat $R$-scheme such that its reduced special fibre is an snc divisor in $\mathcal{X}$ in the sense of Definition 2.3.1. Such schemes $\mathcal{X}$ are called *DNC-models* in [BGS95]. For example, proper and strictly semi-stable $R$-schemes (see Definition 2.3.2) are DNC-models. We denote the generic fibre by $X_\eta$ and the irreducible components of the special fibre (with reduced scheme structure) by $X_i$ with $i \in I$. Since each $X_i$ is a Cartier divisor on $\mathcal{X}$ ($\mathcal{X}$ is regular), there exists a homomorphism

$$\iota_i^*\colon\ \mathrm{CH}_{l+1}(\mathcal{X}) \longrightarrow \mathrm{CH}_l(X_i),$$

where $\iota_i\colon X_i \hookrightarrow \mathcal{X}$ is the natural inclusion, see [Ful98, Proposition 2.6, Section 20.1]. Recall the localization exact sequence from [Ful98, Proposition 1.8]

$$\mathrm{CH}_{l+1}(X) \xrightarrow{\iota_*} \mathrm{CH}_{l+1}(\mathcal{X}) \xrightarrow{j^*} \mathrm{CH}_l(X_\eta) \longrightarrow 0,$$

where $\iota\colon X \hookrightarrow \mathcal{X}$ and $j\colon X_\eta \hookrightarrow \mathcal{X}$ are the natural inclusions. This implies that the homomomorphism

$$\sum_i \iota_i^*\colon\ \mathrm{CH}_{l+1}(\mathcal{X}) \longrightarrow \bigoplus_{i\in I} \mathrm{CH}_l(X_i)$$

induces a homomorphism

$$\mathrm{rsp}\colon\ \mathrm{CH}_l(X_\eta) \longrightarrow \bigoplus_{i\in I} \mathrm{CH}_l(X_i)/\operatorname{Im}\Phi_{\mathcal{X}}, \tag{6.1.1}$$

where $\Phi_{\mathcal{X}}$ is the composition

$$\Phi_{\mathcal{X}}\colon \ \mathrm{CH}_{l+1}(X) \xrightarrow{\iota_*} \mathrm{CH}_{l+1}(\mathcal{X}) \xrightarrow{\sum \iota_i^*} \bigoplus_{i\in I} \mathrm{CH}_l(X_i),$$

see also Definition 3.4.4. We additionally denote the composition with the projection onto the $\mathrm{CH}_l(X_i)$ factor by $\Phi_{\mathcal{X},X_i}$. The morphism $\Phi_{\mathcal{X}}$ has been used for instance in [PS23, Definition 3.1] for $l=0$ to obtain an obstruction to a decomposition of the diagonal of the (geometric) generic fibre, see also (4.3.2).

Since intersecting with divisor is commutative by [Ful98, Corollary 2.4.2], we see that the image of rsp is contained in the subgroup of $\bigoplus_i \mathrm{CH}_l(X_i)/\operatorname{Im}\Phi_{\mathcal{X}}$ consisting of classes of elements $(\alpha_i)_i \in \bigoplus_i \mathrm{CH}_l(X_i)$ which satisfy

$$\alpha_i \cdot (X_i \cap X_j) = \alpha_j \cdot (X_i \cap X_j) \in \mathrm{CH}_{l-1}(X_i \cap X_j).$$

We omit this in the following to simplify notation and the arguments. A version of this subgroup is studied under the name prelog Chow ring in [BBG23, BBG22].

The homomorphism rsp can be thought of as a refined version of Fulton's specialization map: Write $X = \sum_i a_i X_i \in \operatorname{Pic}(\mathcal{X})$ for some non-negative integers $a_i \in \mathbb{Z}_{\geq 0}$. Then for any $(l+1)$-cycle $\tilde{\alpha} \in \mathrm{CH}_{l+1}(\mathcal{X})$ the additivity of the intersection product with divisors yields

$$\iota^*\alpha = X\cdot\alpha = \sum_i a_i(X_i\cdot\alpha) = \sum_i a_i \iota_{X_i,X\,*}\iota_i^*\alpha \in \mathrm{CH}_l(X),$$

where $\iota\colon X \hookrightarrow \mathcal{X}$, $\iota_i\colon X_i \hookrightarrow \mathcal{X}$, and $\iota_{X_i,X}\colon X_i \hookrightarrow X$ are the natural inclusions. In particular, we obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_{l+1}(\mathcal{X}) \arrow[r, "\sum_i \iota_i^*"'] \arrow[d, "j^*"] \arrow[rr, bend left, "\iota^*"] & \bigoplus_i \mathrm{CH}_l(X_i) \arrow[r, "\sum_i a_i \iota_{X_i,X\,*}"'] \arrow[d] & \mathrm{CH}_l(X) \arrow[d, equal] \\
\mathrm{CH}_l(X_\eta) \arrow[r, "\mathrm{rsp}"] \arrow[rr, bend right, "\mathrm{sp}"'] & \bigoplus_i \mathrm{CH}_l(X_i)/\operatorname{Im}\Phi \arrow[r, dashed] & \mathrm{CH}_l(X),
\end{tikzcd}$$

where sp is Fulton's specialization map ([Ful98, §20.3]) the dashed arrow is the induced homomorphism on the quotient; this uses that $\iota^*\iota_* = 0$, as $X$ is a principal Cartier divisor on $\mathcal{X}$. Thus Fulton's specialization map factors through rsp.

The main result of this note is that the refined specialization map respects algebraic equivalence in equicharacteristic 0.

**Proposition 6.1.1** (Corollary 6.5.4)**.** *Let $R$ be a discrete valuation ring with algebraically closed residue field $k$ of characteristic* 0 *such that $R$ is a localization of a finite type $k$-algebra. Let $\mathcal{X}$ be a projective DNC model over $R$ with generic fibre $X_\eta$ and special fibre $X$. Denote the irreducible components (with reduced scheme structure) by $X_i$ with $i \in I$. Then* rsp *induces a morphism on the Chow group of $l$-cycles modulo algebraic equivalence*

$$\mathrm{rsp}\colon A_l(X_\eta) \longrightarrow \bigoplus_{i\in I} A_l(X_i)/\operatorname{Im}\Phi_{\mathcal{X}}.$$

This result may lead to the construction of a rationally connected variety with non-trivial Griffiths group, i.e. an algebraic cycle which is homologous to 0 but not algebraically equivalent to 0. A strategy using degenerations similar to the one in Section 4.4 and [PS23, Section 5] requires a variety with a non-trivial torsion element in the Griffiths group as a base example; such examples have been constructed in the literature, see for instance [Sch92, RS10, Tot16b, Sch25a, Ale23]. Additionally, we need to ensure that the algebraic cycle representing the above torsion element in the Griffiths group lifts to the generic fibre of a suitable degeneration. An extra argument is required to show that the base change of the lift to the geometric generic fibre is not algebraically equivalent to 0; in the case of 1-cycles on rationally connected varieties this is shown in [KT25, Theorem 6].

Throughout this chapter, let $R$ be a discrete valuation ring with perfect residue field $k$ and fraction field $K$. We usually denote an $R$-scheme by calligraphic letters as $\mathcal{X}$, $\mathcal{Y}$, or $\mathcal{Z}$. The special fibre is usually denoted by the corresponding latin letter $X$, $Y$, or $Z$. We use a subscript $\eta$ for the generic fibre, e.g. $X_\eta$, $Y_\eta$, or $Z_\eta$.

We start with a brief recollection on DNC-models from [BGS95] as well as regular embeddings and lci morphisms as used in [Ful98]. Section 6.4 summarizes the required parts of an intersection theory on regular schemes over discrete valuation rings which is developed in the PhD thesis of A. Weber [Web15]. Proposition 6.1.1 is proved in the final section.

## 6.2. DNC models

**Definition 6.2.1** (DNC model, [BGS95])**.** A *DNC model* is a regular proper flat $R$-scheme $\mathcal{X}$ such that the reduced special fibre is an snc divisor in $\mathcal{X}$. If a smooth proper $K$-variety $X$ is isomorphic to $\mathcal{X} \times_R K$, we say that $\mathcal{X}$ is a *DNC model of $X$*.

**Remark 6.2.2.** A strictly semi-stable $R$-scheme as in Definition 2.3.2 is a DNC-model, if it is proper over $R$.

Hironaka's resolution of singularities implies the existence of DNC models, see for instance [BGS95, Section (1.1)]. More precisely, we have the following:

**Proposition 6.2.3.** *Assume that the dvr $R$ is the localization of an algebra of finite type over a field of characteristic* $0$*. Then for any proper flat $R$-scheme $\mathcal{X}$ with smooth generic fibre $X_\eta$, there exists a DNC model $\tilde{\mathcal{X}}$ and a morphism of $R$-schemes $\pi\colon \tilde{\mathcal{X}} \to \mathcal{X}$ such that the induced morphism on the generic fibres $\pi_\eta\colon \tilde{X}_\eta \to X_\eta$ is the identity. If $\mathcal{X}$ is projective over $R$, then so is $\tilde{\mathcal{X}}$.*

*Sketch of proof:* Let $A$ be a finite type $k$-algebra such that $R$ is a localization of $A$. Since $\mathcal{X}$ is of finite type over $R$, there exists an affine open subset $U \subset \operatorname{Spec} A$ such that the composition $\mathcal{X} \to \operatorname{Spec} R \to U$ is of finite type.

By Hironaka's resolution of singularities [Hir64, Main Theorem I, p. 138], there is an integral regular scheme $\mathcal{X}'$ together with a projective morphism $\mathcal{X}' \to \mathcal{X}$. Then [Hir64, Corollary 3, p. 146] implies that there exists a DNC-model $\tilde{\mathcal{X}}$ as claimed.

Since the generic fibre $X_\eta$ is smooth, the sequence of blow-ups used in the resolution of singularities induces an isomorphism on the generic fibre. $\square$

## 6.3. Regular embeddings and lci morphisms

**Definition 6.3.1** ([EGAIV, Définition 16.9.2] and [Ful98, Section B.7])**.**

(a) A closed embedding $\iota\colon X \hookrightarrow Y$ of locally Noetherian schemes is called a *regular embeddding of codimension $p$* if every point $x \in X$ admits an affine open neighbourhood $x \in \operatorname{Spec} A \subset Y$ in $Y$ such that the ideal $I \subset A$ of the closed embedding is generated by a regular sequence $a_1, \ldots, a_d \in A$ of length $d$, i.e. $I = (a_1, \ldots, a_d) \subsetneq A$ is a proper ideal and $a_i \in A/(a_1, \ldots, a_{i-1})$ is not a zero-divisor for each $i \in \{1, \ldots, d\}$.

(b) A morphism $\varphi\colon X \to Z$ of locally Noetherian schemes is called an *locally complete intersection morphism* or short *lci morphism* if it factors as a regular (closed) embedding followed by a smooth morphism.

**Remark 6.3.2.** We use here the convention of [Ful98, Section B.7] that regular embeddings are closed and lci morphism admit a global factorization as above.

The following lemmas provide two examples of lci morphisms.

**Lemma 6.3.3** ([EGAIV, Proposition 19.1.1])**.** *Any closed embedding $\varphi\colon \mathcal{Z} \to \mathcal{X}$ of regular separated scheme of finite type over a locally Noetherian base scheme $S$ is a regular embedding. In particular, any projective morphism of regular separated schemes of finite type over a locally Noetherian base scheme $S$ is a lci morphism.*

**Lemma 6.3.4.** *Let $\varphi\colon \mathcal{Z} \longrightarrow \mathcal{X}$ be a lci morphism between separated scheme of finite type over $R$. Let $D \subset \mathcal{X}$ be a closed subscheme which is also an effective Cartier divisor on $\mathcal{X}$. Assume that $D' := \varphi^{-1}(D)$ is also an effective Cartier divisor in $\mathcal{Z}$. Then the induced morphism*

$$\varphi'\colon D' \longrightarrow D$$

*is also a lci morphism.*

*Proof.* Since $\varphi\colon \mathcal{Z} \longrightarrow \mathcal{X}$ is a lci morphism, we can write $\varphi$ as

$$\mathcal{Z} \xrightarrow{\iota} \mathcal{Y} \xrightarrow{\pi} \mathcal{X},$$

where $\iota$ is a regular embedding and $\pi$ is a closed embedding. Consider the fibre product diagram

$$\begin{tikzcd} D' \arrow[r, "\iota'"] \arrow[d] \arrow[rr, bend left, "\varphi'"] & D \times_{\mathcal{X}} \mathcal{Y} \arrow[r, "\pi'"] \arrow[d] & D \arrow[d] \\ \mathcal{Z} \arrow[r, "\iota"] \arrow[rr, bend right, "\varphi"'] & \mathcal{Y} \arrow[r, "\pi"] & \mathcal{X}. \end{tikzcd}$$

We know that $\pi'$ is smooth morphism and $\iota'$ is a closed embedding by base change. We aim to show that $\iota'$ is also a regular embedding. Since $D \times_{\mathcal{X}} \mathcal{Y} \subset \mathcal{Y}$ is an effective Cartier divisor in $\mathcal{Y}$ by [GW20, Proposition 11.50(a)], we can replace $\mathcal{X}$ with $\mathcal{Y}$ and assume that the morphisms $\pi$ and $\pi'$ are the identity.

Since $D'$ and $D$ are effective Cartier divisors, the natural inclusions $\iota_{D'}\colon D' \hookrightarrow \mathcal{Z}$ and $\iota_D\colon D \hookrightarrow \mathcal{X}$ are regular embeddings. In particular the composition

$$\varphi \circ \iota_{D'} = \iota_D \circ \varphi'\colon D' \longrightarrow \mathcal{X}$$

is a regular embedding and locally we have $N_{\mathcal{X}/D'} \simeq (\iota_{D'})^* N_{\mathcal{X}/\mathcal{Z}} \oplus N_{\mathcal{Z}/D'}$ by [EGAIV, Proposition 19.1.5]. It suffices to show by *loc. cit.* that the sequence of sheaves

$$0 \longrightarrow \varphi'^* N_{\mathcal{X}/D} \longrightarrow N_{\mathcal{X}/D'} \longrightarrow N_{D/D'} \longrightarrow 0$$

is exact and locally split. The sequence is always right exact by [EGAIV, Proposition 16.2.7]. Since $D$ and $D'$ are Cartier divisors, we have

$$\begin{aligned}(\varphi')^* N_{\mathcal{X}/D} &= (\varphi')^* \iota_D^* \mathcal{O}_{\mathcal{X}}(D) = (\iota_{D'})^* \varphi^* \mathcal{O}_{\mathcal{X}}(D) = (\iota_{D'})^* \mathcal{O}_{\mathcal{Z}}(D');\\ N_{\mathcal{Z}/D'} &= (\iota_{D'})^* \mathcal{O}_{\mathcal{Z}}(D').\end{aligned}$$

Hence locally the above sequence reads

$$0 \longrightarrow (\iota_{D'})^* \mathcal{O}_{\mathcal{Z}}(D') \longrightarrow (\iota_{D'})^* N_{\mathcal{X}/\mathcal{Z}} \oplus (\iota_{D'})^* \mathcal{O}_{\mathcal{Z}}(D') \longrightarrow N_{D/D'} \longrightarrow 0,$$

which is clearly (locally) split exact. Hence, $\varphi'$ is a regular embedding. □

## 6.4. Intersection theory on regular schemes over an dvr

Throughout this subsection we will assume that for any integral separated scheme $\mathcal{Z}$ of finite type over $R$, there exists a regular integral scheme $\mathcal{Z}'$ over $R$ together with a proper dominant morphism $\mathcal{Z}' \to \mathcal{Z}$ which is generically an isomorphism. Hence the assumption $A(\operatorname{Spec} R, \mathbb{Z}, \leq n)$ in [Web15, Assumption 3.1.2] is satisfied.

**Example 6.4.1.** The localization of an algebra of finite type over a field of characteristic 0 satisfies the above condition, see Proposition 6.2.3.

The intersection theory in [Web15] is described in terms of bivariant classes.

**Definition 6.4.2** ([Ful98, Definition 17.1])**.** Let $f\colon X \to Y$ be a morphism of schemes over some base scheme $S$. A *bivariant class* $c$ of degree $p$ over $f$ is a collection of homomorphisms

$$c_g^{(l)}\colon \mathrm{CH}_l(Y') \longrightarrow \mathrm{CH}_{l-p}(Y' \times_Y X)$$

for all morphisms $g\colon Y' \to Y$ and all $l$, compatible with proper push-forward, flat pull-back and (refined) Gysin maps, i.e. for every commutative diagram of fibre products squares

$$\begin{array}{ccccc} X'' & \xrightarrow{f''} & Y'' & \xrightarrow{\psi'} & Z'' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_0} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{\psi} & Z' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \\ X & \xrightarrow{f} & Y & & \end{array}$$

with $g\colon Y' \to Y$ an arbitrary morphism the following conditions are satisfied:

(C1) If $h$ is proper (and $\psi = \mathrm{id}_{Y'}$), then we have for all $\alpha \in \mathrm{CH}_l(Y'')$

$$c_g^{(l)}(h_*\alpha) = h'_* c_{gh}^{(l)}(\alpha) \in \mathrm{CH}_{l-p}(X').$$

(C2) If $h$ is flat of relative dimension $n$ (and $\psi = \mathrm{id}_{Y'}$), then for all $\alpha \in \mathrm{CH}_l(Y')$

$$c_{gh}^{(l+n)}(h^*\alpha) = (h')^* c_g^{(l)}(\alpha) \in \mathrm{CH}_{l+n-p}(X'').$$

(C3) If $h_0$ is a regular embedding of codimension $e$ and $\psi$ is an arbitrary morphism, then we have for all $\alpha \in \mathrm{CH}_l(Y')$

$$h_0^! c_g^{(l)}(\alpha) = c_{gh}^{(l-e)}(h_0^!\alpha) \in \mathrm{CH}_{l-p-e}(X'').$$

We denote the group of all bivariant classes of degree $p$ over $f$ by $A^p(f)$.

The most important bivariant classes are given by pullbacks along flat morphisms and (refined) Gysin maps for regular imbeddings or more generally lci morphisms.

**Example 6.4.3.** 1. Any flat morphism $f\colon X \to Y$ of relative dimension $p$ induces a bivariant class $[f] \in A^{-p}(f)$ given by

$$[f]_g^{(l)} := (f')^*\colon \mathrm{CH}_l(Y') \longrightarrow \mathrm{CH}_{l+p}(Y' \times_Y X),$$

where $f'\colon Y' \times_Y X \to Y'$ is the induced flat morphism of relative dimension $p$, see [Ful98, Proposition 1.7, Lemma 1.7.1, Theorem 6.2].

2. Any locally complete intersection morphism $f\colon X \to Y$ of codimension $p$ (e.g. regular imbeddings of codimension $p$) induces a bivariant class $[f] \in A^p(f)$ given by

$$[f]_g^{(l)} := f^!\colon \mathrm{CH}_l(Y') \longrightarrow \mathrm{CH}_{l+p}(Y' \times_Y X),$$

where $f^!$ is the refined Gysin map in [Ful98, §6.6], see [Ful98, Proposition 6.6].

**Remark 6.4.4.** The work of Kleiman-Thorup [KT87, Remarks after Proposition 3.3] shows that for any regular scheme $Y$ over $R$ any morphism $f\colon X \to Y$ of $R$-schemes is equipped with a unique bivariant class over $f$, which we denote by $[f]$, see [Web15, Remark 4.2.4 (i)].

The main result of [Web15] is the existence of an intersection product on regular schemes over a base scheme $S$ under the assumption stated at the beginning of this subsection.

**Theorem 6.4.5** ([Web15, Theorem 4.3.3 and Lemma 4.3.5 (ii)])**.** *Let $\mathcal{X}$ be a regular separated scheme of finite type over $R$. Then there exists an intersection product*

$$\cdot_{\mathcal{X}}\colon \mathrm{CH}_m(\mathcal{X}) \otimes \mathrm{CH}_n(\mathcal{X}) \longrightarrow \mathrm{CH}_{m+n-\dim \mathcal{X}}(\mathcal{X})$$

*satisfying the following properties:*

*(i) The Chow group $\mathrm{CH}^*(\mathcal{X}) \coloneqq \mathrm{CH}_{\dim \mathcal{X}-*}(\mathcal{X})$ becomes a commutative graded ring with $[X] \in \mathrm{CH}^0(X)$ as unit element.*

*(ii) Let $\iota_{\mathcal{V}_i}\colon \mathcal{V}_i \hookrightarrow \mathcal{X}$ be closed subvarieties of dimension $n_i$ for $i = 1, 2$. Then*

$$[\mathcal{V}_1] \cdot_{\mathcal{X}} [\mathcal{V}_2] = \iota_*[\iota_{\mathcal{V}_1}]^{(n_2)}_{\iota_{\mathcal{V}_2}}([\mathcal{V}_2]) \in \mathrm{CH}_{n_1+n_2-\dim \mathcal{X}}(\mathcal{W}_1 \cap \mathcal{W}_2),$$

*where $\iota\colon \mathcal{V}_1 \times_{\mathcal{X}} \mathcal{V}_2 \longrightarrow \mathcal{X}$ is the natural (proper) morphism.*

A. Weber [Web15, Proposition 4.3.7] also constructed a pullback morphism for any morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of separated scheme of finite type over $R$ with $\mathcal{X}$ regular, which agrees with the pullback morphism along flat morphism and the Gysin map for lci morphisms.

**Proposition 6.4.6.** *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be of regular separated scheme of finite type over $R$. Assume that $f$ is a flat morphism of relative dimension $p$ (respectively a lci morphism of codimension $-p$). Then the flat pullback map (respectively the Gysin morphism for lci morphisms)*

$$f^*\colon \mathrm{CH}_n(\mathcal{X}) \longrightarrow \mathrm{CH}_{n+p}(\mathcal{Y})$$

*satisfies for any cycles $\alpha \in \mathrm{CH}_m(\mathcal{X})$ and $\beta \in \mathrm{CH}_n(\mathcal{X})$:*

$$f^*\alpha \cdot_{\mathcal{Y}} f^*\beta = f^*(\alpha \cdot_{\mathcal{X}} \beta) \in \mathrm{CH}_{n+m+p-\dim \mathcal{X}}(\mathcal{Y}).$$

*Moreover, if $f$ is also proper, then for any $\alpha \in \mathrm{CH}_m(\mathcal{X})$ and $\beta \in \mathrm{CH}_n(\mathcal{Y})$ the following holds*

$$f_*\left(f^*\alpha \cdot_{\mathcal{Y}} \beta\right) = \alpha \cdot_{\mathcal{X}} f_*\beta \in \mathrm{CH}_{n+m-\dim \mathcal{X}}(\mathcal{X}).$$

*Proof.* We will show that the pullback morphism stated in [Web15, Proposition 4.3.7 (i)] coincides with the flat pullback (respectively the Gysin map) if $f$ is flat (respectively a lci morphism). Then the statement follows directly from [Web15, Proposition 4.3.7].

Since both homomorphism are linear, it suffices to prove equality for a closed subvariety $\iota_{\mathcal{V}}\colon \mathcal{V} \hookrightarrow \mathcal{X}$. Then the pullback defined in [Web15, Equation (4.12)] is given by

$$\mathrm{CH}_n(\mathcal{X}) \xrightarrow{F_{\mathcal{X}/\mathcal{X}}} A^{\dim \mathcal{X}-n}(\mathrm{id}_{\mathcal{X}}) \xrightarrow{f^*} A^{\dim \mathcal{X}-n}(\mathrm{id}_{\mathcal{Y}}) \xrightarrow{E_{\mathcal{Y}}} \mathrm{CH}_{n+p}(\mathcal{Y}),$$

where $E_{\mathcal{Y}}$ is the evaluation map on $\mathcal{Y}$ (see [Web15, Definition 4.1.12 (iii)]), $F_{\mathcal{X}/\mathcal{X}}$ is the inverse of $E_{\mathcal{X}}$ under the identity morphism $\mathrm{id}_{\mathcal{X}}$ (see [Web15, Equation (4.7)]), and $f^*$ is the natural morphism of bivariant classes (see [Web15, Lemma/Definition 4.1.3]). Thus we find

$$E_Y f^* F_{\mathcal{Y}/\mathcal{Y}}[V] = E_{\mathcal{Y}} f^*[\iota_V] = [\iota_V]_f^{(\dim Y)}([\mathcal{Y}]) = [\iota_V]_f^{(\dim \mathcal{X})}(f^*[\mathcal{X}]) \in \mathrm{CH}_{n+p}(\mathcal{Y}),$$

where the $f^*$ on the right hand side is the flat pullback respectively the Gysin map for Chow groups. Applying the conditions (C2) and (C3) in the definition for bivariant classes in Definition 6.4.2 yields

$$[\iota_V]_f^{(\dim \mathcal{X})}(f^*[\mathcal{X}]) = f^*[\iota_V]_{\mathrm{id}_{\mathcal{X}}}^{(\dim \mathcal{X})}[\mathcal{X}] = f^*[V] \in \mathrm{CH}_{n+p}(\mathcal{Y}).$$

Hence both pullback map agrees, which finishes the proof of the proposition. □

In the following proposition, we check that the above defined intersection product on regular schemes over $R$ agrees with Fulton's intersection product if the regular scheme is defined over the residue field $k$ or the fraction field $K$ of $R$.

**Proposition 6.4.7.** *Let $\mathcal{X}$ be a regular integral separated scheme of finite type over $R$. Assume that the structure map $\mathcal{X} \to \operatorname{Spec} R$ factors either through $\operatorname{Spec} k$ or through $\operatorname{Spec} K$. Then*

$$\alpha \cdot_{\mathcal{X}} \beta = \alpha \cdot \beta \in \mathrm{CH}_{n+m-\dim \mathcal{X}}(\mathcal{X}),$$

*for any $\alpha \in \mathrm{CH}_m(\mathcal{X})$ and $\beta \in \mathrm{CH}_n(\mathcal{X})$, where $\cdot$ is Fulton's intersection product, see [Ful98, §8].*

*Proof.* Let $F \in \{k, K\}$ be either the residue field or the fraction field of $R$, i.e. we have by assumption that $\mathcal{X} \to \operatorname{Spec} F \to \operatorname{Spec} R$ is the structure morphism of the $R$-scheme $\mathcal{X}$. By linearity of both intersection products it suffices to show equality for closed (integral) subvarieties. Let $\iota_V\colon V \hookrightarrow \mathcal{X}$ and $\iota_W\colon W \hookrightarrow \mathcal{X}$ be closed (integral) subvarieties of $\mathcal{X}$ of dimension $m = \dim V$ and $n = \dim W$. Consider the following commutative diagram of Cartesian squares

$$\begin{tikzcd}
V \times_{\mathcal{X}} W \arrow[r] \arrow[d, "\iota''_W"] & V \times_F W \arrow[r] \arrow[d, "\iota'_W"] & W \arrow[d, "\iota_W"] \\
V \arrow[r, "\gamma_{\iota_V}"] \arrow[rr, bend right, "\iota_V"'] & V \times_F \mathcal{X} \arrow[r, "\mathrm{pr}_2"] & \mathcal{X},
\end{tikzcd}$$

where $\gamma_{\iota_V}$ is the graph morphism of $\iota_V$, which is a regular embedding, as $\mathcal{X}$ is regular. Recall that the intersection product for regular schemes over $R$ is given by

$$[V] \cdot_{\mathcal{X}} [W] = (\iota_V \circ \iota''_W)_* [\iota_V]^{(n)}_{\iota_W}([W]) \in \mathrm{CH}_{m+n-\dim X}(X),$$

see Theorem 6.4.5 (ii). By the uniqueness of the bivariant class for regular schemes (see Remark 6.4.4), we find that the bivariant class

$$[\iota_V] = [\gamma_{\iota_V}][\mathrm{pr}_2] \in A^{m-\dim \mathcal{X}}(\iota_V).$$

Hence, we obtain using the description of the bivariant classes in Example 6.4.3

$$[V]\cdot_{\mathcal{X}}[W] = (\iota_V \circ \iota''_W)_*[\iota_V]^{(n)}_{\iota_W}([W]) = (\iota_V \circ \iota''_W)_*[\gamma_{\iota_V}]^{(m+n)}_{\iota'_W}[\mathrm{pr}_2]^{(n)}_{\iota_W}([W]) = (\iota_V \circ \iota''_W)_*\gamma^!_{\iota_V}([V\times W])$$

in $\mathrm{CH}_{n+m-\dim \mathcal{X}}(\mathcal{X})$. Applying [Ful98, Example 8.1.8] with $\delta\colon \mathcal{X} \to \mathcal{X} \times_F \mathcal{X}$ the diagonal embedding yields

$$[V]\cdot_{\mathcal{X}}[W] = (\iota_V \circ \iota''_W)_*\gamma^!_{\iota_V}([V\times W]) = (\iota_V \circ \iota''_W)_*\delta^!([V]\times[W]) = [V]\cdot[W] \in \mathrm{CH}_{n+m-\dim \mathcal{X}}(\mathcal{X}),$$

which concludes the proof of the proposition. □

## 6.5. Algebraic equivalence specializes

Informally speaking, we aim to show that the refined specialization map defined in (6.1.1) commutes with action of correspondences, see (6.5.1) below for a more precise statement.

Assume that $R$ is a discrete valuation ring with algebraically closed residue field $k$ of characteristic 0 such that $R$ is the localization of a finite type $k$-algebra as in Example 6.4.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be projective DNC models over $R$ as in Definition 6.2.1; denote the irreducible components (with reduced scheme of structure) of the special fibres $X$ and $Y$ by $X_i$ with $i \in I$ and $Y_j$ with $j \in J$, respectively. We denote the generic fibres by $X_\eta$ and $Y_\eta$, respectively.

**Theorem 6.5.1.** *Let the notation be as above. Let $\Gamma \in \mathrm{CH}_m(X_\eta \times Y_\eta)$ be an $m$-cycle. Then there exist $m$-cycles $\Gamma_{i,j} \in \mathrm{CH}_m(X_i \times_k Y_j)$ for each $(i,j) \in I \times J$ (see (6.5.2) below for explicit description) such that the following holds:*

*(a) Let $\gamma \in \mathrm{CH}_{n+1}(X)$ be an $n+1$-cycle. Then we have*

$$\left(\sum_{i\in I} \Gamma_{i,j\,*}\Phi_{\mathcal{X},X_i}(\gamma)\right)_j \in \operatorname{Im}\Phi_{\mathcal{Y}}.$$

*Hence $\sum_{i\in I}\Gamma_{i,j\,*}$ induces a well-defined homomorphism on the image of* rsp.

*(b) For every $n$-cycle $\alpha \in \mathrm{CH}_n(X_\eta)$, we have*

$$\operatorname{rsp}(\Gamma_*\alpha) = \left(\sum_{i\in I}\Gamma_{i,j\,*}\operatorname{rsp}(\alpha)\right)_j \in \bigoplus_{j\in J}\mathrm{CH}_{m+n-\dim X_\eta}(Y_j)/\operatorname{Im}\Phi_{\mathcal{Y}} \tag{6.5.1}$$

**Remark 6.5.2.** If $\mathcal{X}$ and $\mathcal{Y}$ are smooth projective $R$-schemes, then $\Gamma_{0,0} \in \mathrm{CH}_m(X\times_k Y)$ is the specialization of the $m$-cycle $\Gamma \in \mathrm{CH}_m(X_\eta \times_K Y_\eta)$ on the generic fibre.

*Proof.* We give an explicit way to construct the cycles $\Gamma_{i,j}$ and check that the constructed cycles satisfy the desired properties.

**Step 1: Construction of $\Gamma_{i,j}$** Let $\pi\colon \mathcal{Z} \longrightarrow \mathcal{X}\times_R \mathcal{Y}$ be a resolution as in Proposition 6.2.3 and let $Z_l$ with $l \in L$ denote the irreducible components of the special fibre $Z = \mathcal{Z}\times_R k$ with reduced scheme structure.

For $j \in J$, consider the pullback of the effective Cartier divisor $X\times_k Y_j \subset \mathcal{X}\times_R\mathcal{Y}$ along $\pi$, which is an effective Cartier divisor whose supported lies in $Z$, in particular we can write

$$\pi^*(X\times_k Y_j) = \sum_{l\in L_j} a_{jl}Z_l$$

for some non-empty subset $L_j \subset L$ and some positive integers $a_{jl} \in \mathbb{Z}_{>0}$. For each $Z_l$ with $l \in L_j$, there exists an $i(j,l) \in I$ (not necessarily unique!) such that $\pi(Z_l) \subset X_{i(j,l)} \times_k Y_j$.

Let $j_Z\colon Z_\eta \hookrightarrow \mathcal{Z}$ and $j_X\colon X_\eta \hookrightarrow \mathcal{X}$ be the natural inclusions of the generic fibres. Then the localization exact sequence [Ful98, Proposition 1.8] shows that the pullback maps $j_Z^*$ and $j_X^*$ on Chow groups are surjective. Let $\tilde{\Gamma} \in \mathrm{CH}_{m+1}(\mathcal{Z})$ and $\tilde{\alpha} \in \mathrm{CH}_{n+1}(\mathcal{X})$ be algebraic cycles such that $j_Z^*\tilde{\Gamma} = \Gamma \in \mathrm{CH}_m(Z_\eta) = \mathrm{CH}_m(X_\eta\times_K Y_\eta)$ and $j_X^*\tilde{\alpha} = \alpha \in \mathrm{CH}_n(X_\eta)$.

We denote the projection onto the first and second factor by $p\colon \mathcal{X} \times_R \mathcal{Y} \to \mathcal{X}$ and $q\colon \mathcal{X} \times_R \mathcal{Y} \to \mathcal{Y}$, respectively. Since $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are regular schemes, the projective morphism $p \circ \pi\colon \mathcal{Z} \to \mathcal{X}$ and $q \circ \pi\colon \mathcal{Z} \to \mathcal{Y}$ are lci morphisms by Lemma 6.3.3. By Theorem 6.4.5, there exists a well-defined cycle

$$\tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \tilde{\alpha} \in \mathrm{CH}_{m+n-\dim X_\eta+1}(\mathcal{Z}).$$

The projection formula for the intersection with (pseudo-)divisors [Ful98, Proposition 2.3 (c)] and the definition of the Gysin map for effective Cartier divisors [Ful98, Proposition 2.6] yields

$$\begin{aligned}
&\iota_{Y_j}^* q_* \pi_* \left(\tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \tilde{\alpha}\right) \\
&= [Y_j] \cdot q_* \pi_* \left(\tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \tilde{\alpha}\right) \\
&= q_* \left([X \times_k Y_j] \cdot \pi_* \left(\tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \tilde{\alpha}\right)\right) \\
&= q_* \pi_* \left(\sum_{l \in L_j} a_{jl} Z_l \cdot \left(\tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \tilde{\alpha}\right)\right) \\
&= q_* \pi_* \left(\sum_{l \in L_j} a_{jl} \iota_{Z_l}^* \left(\tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \tilde{\alpha}\right)\right) \in \mathrm{CH}_{m+n+\dim X_\eta}(Y_j),
\end{aligned}$$

where $\cdot$ means here the intersection with the Cartier divisors as defined in [Ful98, Section 2.3]. Since $\iota_{Z_l}$ is a regular embedding, Propositions 6.4.6 and 6.4.7 imply

$$\iota_{Y_j}^* q_* \pi_* \left(\tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \tilde{\alpha}\right) = q_* \pi_* \left(\sum_{l \in L_j} a_{jl} \left(\iota_{Z_l}^* \tilde{\Gamma} \cdot_{Z_l} \iota_{Z_l}^* (p \circ \pi)^* \tilde{\alpha}\right)\right) \in \mathrm{CH}_{m+n+\dim X_\eta}(Y_j).$$

Applying [Ful98, Theorem 6.5, Proposition 6.6 (c)] to the obvious commuting diagram shows

$$\begin{aligned}
&q_* \pi_* \sum_{l \in L_j} a_{jl} \left(\iota_{Z_l}^* \tilde{\Gamma} \cdot_{Z_l} \iota_{Z_l}^* (p \circ \pi)^* \tilde{\alpha}\right) \\
&= q_* \sum_{l \in L_j} a_{jl} \pi_* \left(\iota_{Z_l}^* \tilde{\Gamma} \cdot_{Z_l} \pi^* p^* \iota_{X_{i(j,l)}}^* \tilde{\alpha}\right) \\
&= q_* \sum_{l \in L_j} a_{jl} \left(\pi_* \iota_{Z_l}^* \tilde{\Gamma} \cdot_{X_{i(j,l)} \times_k Y_j} p^* \iota_{X_{i(j,l)}}^* \tilde{\alpha}\right) \\
&= \sum_{i \in I} q_* (\pi_* \sum_{\substack{l \in L_j \\ i(j,l)=i}} a_{jl} \iota_{Z_l}^* \tilde{\Gamma} \cdot_{X_i \times_k Y_j} p^* \iota_{X_i}^* \tilde{\alpha}) \in \mathrm{CH}_{m+n+\dim X_\eta}(Y_j).
\end{aligned}$$

We set

$$\Gamma_{i,j} \coloneqq \sum_{\substack{l \in L_j \\ i(j,l)=i}} a_{jl} \pi_* \iota_{Z_l}^* \tilde{\Gamma} \in \mathrm{CH}_m(X_i \times_k Y_j). \tag{6.5.2}$$

Then the above computation shows that

$$\iota_{Y_j}^* q_* \pi_* \left( \tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \tilde{\alpha} \right) = \sum_{i \in I} q_* \left( \Gamma_{i,j} \cdot_{X_i \times_k Y_j} p^* \iota_{X_i}^* \tilde{\alpha} \right) \tag{6.5.3}$$

in $\mathrm{CH}_{m+n+\dim X_\eta}(Y_j)$.

**Step 2: Proof of the properties** Let $\gamma \in \mathrm{CH}_{n+1}(X)$ be an $(n+1)$-cycle. Since the natural pushforward map

$$\bigoplus_{i \in I} \mathrm{CH}_{n+1}(X_i) \longrightarrow \mathrm{CH}_{n+1}(X)$$

is surjective (it is already surjective on the level of algebraic cycles) and the homomorphism $\Gamma_{i,j\,*} \Phi_{\mathcal{X}}$ is clearly linear, we can assume that $\gamma \in \mathrm{CH}_{n+1}(X_{i_0})$ for some $i_0 \in I$. For $j \in J$ arbitrary, we have

$$\sum_{i \in I} \Gamma_{i,j\,*} \Phi_{\mathcal{X},X_i}(\gamma) = \sum_{i \in I} q_* \left( \Gamma_{i,j} \cdot_{X_i \times_k Y_j} p^* \iota_{X_i}^* \iota_{X_{i_0}\,*} \gamma \right) \overset{(6.5.3)}{=} \iota_{Y_j}^* q_* \pi_* \left( \tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \iota_{X_{i_0}\,*} \gamma \right)$$

in $\mathrm{CH}_{m+n+\dim X_\eta}(Y_j)$. Consider now the fibre square

$$\begin{tikzcd}
X_{i_0} \times_{\mathcal{X}} \mathcal{Z} \arrow[r, "\iota'_{X_{i_0}}"'] \arrow[d] & \mathcal{Z} \arrow[d, "p \circ \pi"] \\
X_{i_0} \arrow[r, "\iota_{X_{i_0}}"] & \mathcal{X}
\end{tikzcd}$$

The horizontal arrows are closed embeddings of effective Cartier divisors (hence regular embedddings) and the vertical arrows are lci morphisms, see in particular Lemma 6.3.4. Thus [Ful98, Theorem 6.2 and Proposition 6.6] implies that

$$\iota_{Y_j}^* q_* \pi_* \left( \tilde{\Gamma} \cdot_{\mathcal{Z}} (p \circ \pi)^* \iota_{X_{i_0}\,*} \gamma \right) = \iota_{Y_j}^* q_* \pi_* \left( \tilde{\Gamma} \cdot_{\mathcal{Z}} \iota'_{X_{i_0}\,*} (p \circ \pi)^* \gamma \right) \tag{6.5.4}$$

We aim to apply the projection formula from Proposition 6.4.6, but the problem is that $X_{i_0} \times_{\mathcal{X}} \mathcal{Z}$ is in general not regular (not even reduced). To fix this, we write the effective Cartier divisor $X_{i_0} \times_{\mathcal{X}} \mathcal{Z}$ in $\mathcal{Z}$ as

$$X_{i_0} \times_{\mathcal{X}} \mathcal{Z} = \sum_{l \in L} b_l Z_l,$$

for some non-negative integers $b_l \in \mathbb{Z}_{\geq 0}$ (not all equal to zero). Since $X_{i_0}$ is regular and defined over a perfect field $k$, it is smooth over $k$, see [GW20, Remark 6.33.]. Hence, [Ful98, Proposition 8.12 (b)] shows that

$$\iota'_{X_{i_0}*}(p\circ\pi)^*\gamma = \iota'_{X_{i_0}*}\left([D']\cdot_{(p\circ\pi)}\gamma\right) = \iota'_{X_{i_0}*}\left(\sum_l b_l[Z_l]\cdot_{(p\circ\pi)}\gamma\right) = \sum_l b_l \iota_{Z_l*}(p\circ\pi)^*\gamma$$

in $\mathrm{CH}_{n+1+\dim\mathcal{Z}-\dim\mathcal{X}}(\mathcal{Z})$ where $\iota_{Z_l}\colon Z_l \hookrightarrow \mathcal{Z}$ is the natural inclusion and we denote the map $Z_l \to X_{i_0}\times_{\mathcal{X}}\mathcal{Z} \to X_{i_0}$ for simplicity also by $p\circ\pi$. Note that it is a projective morphism between regular schemes and hence lci, see Lemma 6.3.3. Hence the equation (6.5.4) can be rewritten as

$$\sum_l b_l \iota^*_{Y_j} q_*\pi_*\left(\tilde{\Gamma}\cdot_{\mathcal{Z}}(p\circ\pi)^*\iota_{X_{i_0}*}\gamma\right) = \iota^*_{Y_j} q_*\pi_*\left(\tilde{\Gamma}\cdot_{\mathcal{Z}}\iota_{Z_l*}(p\circ\pi)^*\gamma\right)$$

As $Z_l$ is regular scheme, we can apply the projection formula from Proposition 6.4.6 (and use Proposition 6.4.7) to get for all $l\in L$

$$\iota^*_{Y_j} q_*\pi_*\left(\tilde{\Gamma}\cdot_{\mathcal{Z}}\iota_{Z_l*}(p\circ\pi)^*\gamma\right) = \iota^*_{Y_j} q_*\pi_*\iota_{Z_l*}\left(\iota^*_{Z_l}\tilde{\Gamma}\cdot_{Z_l}(p\circ\pi)^*\gamma\right) = \iota^*_{Y_j}\iota_{Y*}q_*\pi_*\left(\iota^*_{Z_l}\tilde{\Gamma}\cdot_{Z_l}(p\circ\pi)^*\gamma\right)$$

$\mathrm{CH}_{m+n+\dim X_\eta}(Y_j)$, where we use in the last equality the functioriality of proper pushforward and that the diagram

$$\begin{array}{ccc} Z_l & \xrightarrow{\iota_{Z_l}} & \mathcal{Z} \\ \downarrow{\scriptstyle q\circ\pi} & & \downarrow{\scriptstyle q\circ\pi} \\ Y & \xrightarrow{\iota_Y} & \mathcal{Y} \end{array}$$

commutes. Hence, we proved for every $j\in J$ that

$$\sum_{i\in I}\Gamma_{i,j*}\Phi_{\mathcal{X},X_i}(\gamma) = \iota^*_{Y_j}\iota_{Y*}\left(\sum_l b_l q_*\pi_*\left(\iota^*_{Z_l}\tilde{\Gamma}\cdot_{Z_l}(p\circ\pi)^*\gamma\right)\right) \in \mathrm{CH}_{m+n+\dim X_\eta}(Y_j),$$

in particular it is contained in the image of $\Phi_{\mathcal{Y}}$, which finishes the proof of (a).

Part (b) follows directly from the construction of the $\Gamma_{i,j}$ in (6.5.2) and part (a). Proposition 6.4.6 applied to the embedding $j_Z\colon Z_\eta \hookrightarrow \mathcal{Z}$ shows together with Proposition 6.4.7 that

$$j_Z^*(\tilde{\Gamma}\cdot_{\mathcal{Z}}(p\circ\pi)^*\tilde{\alpha}) = j_Z^*\tilde{\Gamma}\cdot_{Z_\eta} j_Z^*(p\circ\pi)^*\tilde{\alpha} = \Gamma\cdot_{X_\eta\times_K Y_\eta} p^*\alpha \in \mathrm{CH}_{m+n-\dim X_\eta}(X_\eta\times Y_\eta).$$

In particular we find that

$$q_*\pi_*\left(\tilde{\Gamma}\cdot_{\mathcal{Z}}(p\circ\pi)^*\tilde{\alpha}\right)\in \mathrm{CH}_{m+n+\dim\mathcal{Y}}(\mathcal{Y})$$

restricts to $\Gamma_*\alpha$ on $Y_\eta$. Hence $\mathrm{rsp}(\Gamma_*\alpha)$ is given by (6.5.3) (in fact its image under the quotient modulo $\operatorname{Im}\Phi_{\mathcal{Y}}$) proved in Step 1, which shows (6.5.1) and thus (b). □

**Remark 6.5.3.** The construction of $\Gamma_{i,j}$ in (6.5.2) depends on several choice, namely the choice of a resolution $\mathcal{Z}$ of $\mathcal{X}\times_R\mathcal{Y}$, the choice of a element $\tilde{\Gamma}\in\mathrm{CH}_{m+1}(\mathcal{Z})$ whose restriction to the generic fibre is $\Gamma$, and the choice of an index $i(j,l)\in I$ such that the projection of $Z_l$ is contained in $X_{i(j,l)}\times Y_j$. In particular, the cycle $\Gamma_{i,j}$ is in general not unique. One can show that the induced homomorphism

$$(\Gamma_{i,j\,*})_{i,j}:\bigoplus_{i\in I}\mathrm{CH}_n(X_i)/\operatorname{Im}\Phi_{\mathcal{X}}\longrightarrow\bigoplus_{j\in J}\mathrm{CH}_{l+m-\dim X_\eta}(Y_j)/\operatorname{Im}\Phi_{\mathcal{Y}},$$

is independent of these choices, when restricted to elements $(\alpha_i)_i\in\bigoplus_{i\in I}\mathrm{CH}_n(X_i)$ such that $\alpha_i\cdot_{X_i}X_i\cap X_{i'}=\alpha_{i'}\cdot_{X_{i'}}X_i\cap X_{i'}$ for all $i,i'\in I$.

**Corollary 6.5.4.** *Let $R$ be a discrete valuation ring with algebraically closed residue field $k$ of characteristic* 0 *such that $R$ is a localization of a finite type $k$-algebra. Let $\mathcal{X}$ be a projective DNC model over $R$ with generic fibre $X_\eta$ and special fibre $X$. Denote the irreducible components (with reduced scheme structure) by $X_i$ with $i\in I$. Then* rsp *induces a morphism on the Chow group of $l$-cycles modulo algebraic equivalence*

$$\mathrm{rsp}\colon A_l(X_\eta)\longrightarrow\bigoplus_{i\in I}A_l(X_i)/\operatorname{Im}\Phi_{\mathcal{X}}.$$

*Proof.* It suffices to check that $l$-cycles which are algebraically equivalent to 0 are mapped to $l$-cycles which are algebraically equivalent to 0. By definition (see Section 1.6) $\gamma\in\mathrm{CH}_l(X_\eta)$ is algebraically equivalent to 0, if there exists a curve $C$ and an $(l+1)$-cycle on $\Gamma\in\mathrm{CH}_{l+1}(C\times_K X_\eta)$ such that $\gamma=\Gamma_*(c_0-c_1)$ for two closed points $c_0,c_1\in C$. Without loss of generality we can assume that $C$ is a smooth projective integral curve, see [ACMV17].

By taking the closure of $C\subset\mathbb{P}^N_K$ in $\mathbb{P}^N_R$ and using resolution of singularities, more precisely Proposition 6.2.3, we can assume that there exists a projective DNC-model $\mathcal{C}$ of $C$ over $R$. Then the corollary follows directly from Theorem 6.5.1, more precisely (6.5.1). □

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# Urheberschaftserklärung

Ich erkläre hiermit, dass diese Arbeit ausschließlich von mir verfasst wurde und dass sie mein eigenes Werk darstellt, sofern im Text nicht anders angegeben.

Ich bestätige, dass alle Zitate, Argumente oder Konzepte, die von anderen Autoren entwickelt wurden, sowie alle Informationsquellen in der gesamten Arbeit referenziert sind.

Diese Arbeit wurde bisher nicht im Rahmen eines anderen Studienabschlusses eingereicht.

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- Jan Lange, *On zero-cycles of varieties over Laurent fields*, arXiv preprint 2402.14406, submitted.
- Jan Lange and Bjørn Skauli, *The Diagonal of (3,3) fivefolds*, Ann. Inst. Fourier, Online first, 31 p., DOI: 10.5802/aif.3735, arXiv: 2303.00562.